

Generating Lie and gauge free differential (super)algebras by expanding Maurer-Cartan forms and Chern-Simons supergravity

J. A. de Azcárraga^{a,1}, J. M. Izquierdo^{b,1}, M. Picón^{a,1} and O. Varela^{a,1}

*a Departamento de Física Teórica, Facultad de Física, Universidad de Valencia
and IFIC, Centro Mixto Universidad de Valencia-CSIC,
E-46100 Burjassot (Valencia), Spain*

*b Departamento de Física Teórica, Universidad de Valladolid
E-47011 Valladolid, Spain*

Abstract

We study how to generate new Lie algebras $\mathcal{G}(N_0, \dots, N_p, \dots, N_n)$ from a given one \mathcal{G} . The (order by order) method consists in expanding its Maurer-Cartan one-forms in powers of a real parameter λ which rescales the coordinates of the Lie (super)group G , $g^{i_p} \rightarrow \lambda^{p_i} g^{i_p}$, in a way subordinated to the splitting of \mathcal{G} as a sum $V_0 \oplus \dots \oplus V_p \oplus \dots \oplus V_n$ of vector subspaces. We also show that, under certain conditions, one of the obtained algebras may correspond to a generalized İnönü-Wigner contraction in the sense of Weimar-Woods, but not in general. The method is used to derive the M-theory superalgebra, including its Lorentz part, from $osp(1|32)$. It is also extended to include gauge free differential (super)algebras and Chern-Simons theories, and then applied to $D = 3$ CS supergravity.

1 Introduction and motivation: four methods to derive new Lie algebras from given ones

The relation of given Lie algebras (and groups) among themselves, and specially the derivation of new algebras from them, is a problem of great interest in mathematics and physics, where it goes back to the old problem of mixing of symmetries and to the advent of supersymmetry itself, the only non-trivial way of enlarging spacetime symmetries (see, respectively, [1] and [2] and the papers reprinted therein). Setting aside the trivial problem of finding whether a Lie algebra is a subalgebra of another one there are, essentially, three different ways of relating and/or obtaining new algebras from given ones.

The *first* one is the *contraction* procedure [3, 4, 5]. In its İnönü and Wigner (IW) simple form [4], the contraction \mathcal{G}_c of a Lie algebra \mathcal{G} is performed with respect to a subalgebra \mathcal{L}_0 by rescaling the basis generators of the coset $\mathcal{G}/\mathcal{L}_0$ by means of a parameter, and then by taking a singular limit for this parameter. The generators in $\mathcal{G}/\mathcal{L}_0$ become abelian in the contracted algebra \mathcal{G}_c , and the subalgebra $\mathcal{L}_0 \subset \mathcal{G}_c$ acts on them. As a result, \mathcal{G}_c has a semidirect structure, and the abelian generators determine an ideal of \mathcal{G}_c ;

¹j.a.de.azcarraga@ific.uv.es, izquierdo@fta.uva.es, moises.picon@ific.uv.es, oscar.varela@ific.uv.es

obviously, \mathcal{G}_c has the same dimension as \mathcal{G} . The contraction process has well known physical applications as *e.g.*, in understanding the non-relativistic limit from a group theoretical point of view, or to explain the appearance of dimensionful generators when the original algebra \mathcal{G} is semisimple (and hence with dimensionless generators). This is achieved by using a dimensionful contraction parameter, as in the derivation of the Poincaré group from the de Sitter groups (there, the parameter is the radius R of the universe, and the limit is $R \rightarrow \infty$). There have been many discussions and variations of the IW contraction procedure (see [6, 7, 8, 9, 10, 11] to name a few), but all of them have in common that \mathcal{G} and \mathcal{G}_c have, necessarily, the same dimension as vector spaces. The contraction process has also been considered for ‘quantum’ algebras (see *e.g.*, [12]).

The *second* procedure is the *deformation* of algebras, and Lie algebras in particular [13, 14, 15, 16] (see also [17, 18, 19]), which allows us to obtain algebras *close*, but not isomorphic, to a given one. This leads to the important notion of rigidity [13, 14, 16] (or physical stability): an algebra is called *rigid* when any attempt to deform it leads to an equivalent (isomorphic) one. From a physical point of view, the deformation process is essentially the inverse to the contraction one (see [17] and the second ref. in [11]), and the dimensions of the original and deformed Lie algebras are again the same. For instance, the Poincaré algebra is not rigid, but the de Sitter algebras, being semisimple, have trivial second cohomology group by the Whitehead lemma and, as a result, they are rigid. One may also consider the Poincaré algebra as a deformation of the Galilei algebra, so that this deformation may be read as a group theoretical prediction of relativity. Thus, the mathematical deformation may be physically considered as a tool for developing a physical theory from another pre-existing one. Quantization itself may also be looked at as a deformation (see [20, 21, 22]), the classical limit being the contraction limit $\hbar \rightarrow 0$.

A *third* procedure to obtain new Lie algebras is the *extension* $\tilde{\mathcal{G}}$ of an algebra \mathcal{G} by another one \mathcal{A} (for details and references see *e.g.* [23]). The extended algebra $\tilde{\mathcal{G}}$ contains \mathcal{A} as an ideal and $\tilde{\mathcal{G}}/\mathcal{A} \approx \mathcal{G}$, but \mathcal{G} is not necessarily a subalgebra of $\tilde{\mathcal{G}}$. The data of the extension problem is \mathcal{G} , \mathcal{A} and an action of \mathcal{G} on \mathcal{A} . When \mathcal{A} is abelian the problem always has a solution, the semidirect sum $\tilde{\mathcal{G}} = \mathcal{A} \ltimes \mathcal{G}$ (in which case \mathcal{G} is a subalgebra of $\tilde{\mathcal{G}}$), but in the general case there may be an obstruction to the extension. Since, for an extension $\tilde{\mathcal{G}}$, $\tilde{\mathcal{G}}/\mathcal{A} \approx \mathcal{G}$ always, $\dim \tilde{\mathcal{G}} = \dim \mathcal{G} + \dim \mathcal{A}$. Thus, once more, the dimension of the resulting algebra is equal to the total number of generators in the algebras involved in obtaining the new one (here, the extension $\tilde{\mathcal{G}}$).

The extension and deformation procedure are both directly governed by various aspects of the cohomology of Lie algebras. For instance, the existence of non-trivial *central* extensions of \mathcal{G} by an abelian algebra \mathcal{A} depends solely on the non-triviality of the second cohomology group $H_0^2(\mathcal{G}, \mathcal{A})$. In this case there exist non-trivial two-cocycles (the constant that accompanies them may also play a dimensionally fundamental role). But cohomology also plays a subtle role in the case of the contraction process since, in general, the contraction generates cohomology. This explains *e.g.*, how the 11-dimensional Galilei group (which is a non-trivial central extension of the ordinary Galilei group by $U(1)$) may be obtained from the *direct* product of the Poincaré group by $U(1)$. This is possible because central extension two-coboundaries, which correspond to trivial, direct products, may become non-trivial two-cocycles in the contraction limit [5, 24]; other examples of this mechanism will be mentioned in Sec. 8. In the case of deformations, a sufficient condition for the rigidity of \mathcal{G} is the vanishing of the second cohomology group, $H^2(\mathcal{G}, \mathcal{G}) = 0$ (hence, all semisimple algebras are rigid). The elements of $H^2(\mathcal{G}, \mathcal{G}) \neq 0$ describe infinitesimal (first order) defor-

mations, which are integrable into a one-parameter family of deformations if $H^3(\mathcal{G}, \mathcal{G}) = 0$ [13, 15].

These procedures can be extended to *super* or \mathbb{Z}_2 -graded Lie algebras, with even (bosonic) and odd (fermionic) generators, by taking into account the specific properties of the Grassmann variables. The cohomology of superalgebras was briefly discussed first in [25] (see also [26, 27, 28]), and is especially important in the context of supersymmetric theories (recent papers on superalgebra extensions are [28, 30, 31]). Deformations of superalgebras have also been considered (see *e.g.*, [29, 28]); for instance, it may be seen that $osp(1|4)$ is the only deformation of the $D=4$, $N=1$ superPoincaré algebra [29]. One of the reasons for the interest of superalgebra cohomology is its relevance in the construction of actions for supersymmetric extended objects. In particular, the generalization to superalgebras of the Chevalley-Eilenberg approach [32] to Lie algebra cohomology is especially important in the construction of the Wess-Zumino (WZ) terms that appear in the superbrane actions [33]. These terms may be shown to be related to extensions of supersymmetry in various spacetime dimensions. Indeed, it may well be that a better description of supersymmetric extended objects requires that ordinary superspace be enlarged with additional coordinates (beyond the standard (x, θ) ones) following a fields/extended superspace variables democracy principle (see [34] and references therein). In fact, many of the spacetime supersymmetry algebras (as the ‘M-algebra’ [35][36, 37, 38, 39]), and their associated enlarged superspaces, may be considered as algebra/group extensions (see [34] and references therein), containing central (and non-central) generators. On the deformation side, one may also apply the algebraic rigidity criterion to superspace [40], since it is given by a group extension [41] and some extensions may also be viewed as examples of deformations.

Nevertheless, we may ask ourselves whether the three above procedures are all that may be useful in finding and discussing the underlying symmetry structure of supersymmetric theories and their interrelations, particularly when these include non-flat geometries as *AdS* ones. Motivated by these considerations, we want to explore in this paper another way to obtain new algebras of increasingly higher dimensions from a given one \mathcal{G} . The idea, originally considered by Hatsuda and Sakaguchi in [42] in a less general context, consists in looking at the algebra \mathcal{G} as described by the Maurer-Cartan (MC) forms on the manifold of its associated group G and, after rescaling some of the group parameters by a factor λ , in expanding the MC forms as series in λ . We shall study the method here in general, and discuss how it can be applied when the rescaling is subordinated to a splitting of \mathcal{G} into a sum of vector subspaces. The resulting *fourth* procedure, the *expansion* method to be described below, is different from the three above albeit, when the algebra dimension does not change in the process, it may lead to a simple IW or IW-generalized contraction (*i.e.*, one that rescales the algebra generators using different powers of λ) in the sense of Weimar-Woods [11], but not always. Furthermore, the algebras to which it leads have in general higher dimension than the original one, in which case they cannot be related to it by any contraction or deformation process. We shall term them *expanded algebras*.

The use of the MC forms to discuss new algebras and superalgebras is specially convenient. It allows us to treat Lie (super)algebras as a particular case of free differential algebras [43, 44, 27] and, from a physical point of view, to have ready the invariant forms that are used to construct actions. In particular, the MC forms on superspace, either enlarged or not, are essential in the formulation of the actions for supersymmetric objects as already mentioned. In fact, there have been indications that the method below may be

used [42, 45] in the construction, starting from the superalgebra of the AdS superstring [46], of a Lie superalgebra that is realized in the context of the IIB D-string. The discussion may also be relevant when considering the structure of all possible enlarged new superspaces, and in particular to see whether the method of [34], which corresponds to the third procedure above (extension of superalgebras) exhausts the number of possible, physically relevant, superspaces. As one might expect, we shall conclude that the extension method leads to a greater variety of (super)algebras and that the expansion procedure may be useful to find and relate existing theories.

In this paper we shall restrict ourselves to giving the general structure of the expansion method (Secs. 2-5) plus some immediate applications concerning the M-theory superalgebra (Sec. 6), gauge free differential (super)algebras (Sec. 7) and Chern-Simons supergravity in $D = 3$ (Sec. 8), leaving further developments for a forthcoming paper.

2 Rescaling of the group parameters and the expansion method

Let G be a Lie group, of local coordinates g^i , $i = 1, \dots, r = \dim G$. Let \mathcal{G} be its Lie algebra² of basis $\{X_i\}$, which may be realized by left-invariant (LI) generators $X_i(g)$ on the group manifold. Let \mathcal{G}^* be the coalgebra, and let $\{\omega^i(g)\}$, $i = 1, \dots, r = \dim G$ be the basis determined by the (dual, LI) Maurer-Cartan (MC) one-forms on G . Then, when $[X_i, X_j] = c_{ij}^k X_k$, the MC equations read

$$d\omega^k(g) = -\frac{1}{2}c_{ij}^k \omega^i(g) \wedge \omega^j(g), \quad i, j, k = 1, \dots, r \quad . \quad (2.1)$$

We wish to show in this section how we may obtain new algebras by means of a redefinition $g^l \rightarrow \lambda g^l$ of some of the group parameters and by looking at the power series expansion in λ of the resulting one-forms $\omega^i(g, \lambda)$. Let θ be the LI canonical form on G ,

$$\theta(g) = g^{-1} dg = e^{-g^i X_i} dg e^{g^i X_i} \equiv \omega^i X_i \quad . \quad (2.2)$$

Since

$$\begin{aligned} e^{-A} de^A &= dA + \frac{1}{2}[dA, A] + \frac{1}{3!}[[dA, A], A] + \frac{1}{4!}[[[dA, A], A], A] + \dots \\ &= dA + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} [\cdot^n \cdot [dA, A], \dots, A], A] \quad , \end{aligned} \quad (2.3)$$

one obtains, for $A \equiv g^k X_k$, $dA = (dg^j)X_j$, the expansion of $\theta(g)$ and of the MC forms $\omega^i(g)$ as polynomials in the group coordinates g^i :

$$\theta(g) = \left[\delta_j^i + \frac{1}{2!}c_{jk}^i g^k + \frac{1}{3!}c_{jk_1}^{h_1} c_{h_1 k_2}^i g^{k_1} g^{k_2} + \frac{1}{4!}c_{jk_1}^{h_1} c_{h_1 k_2}^{h_2} c_{h_2 k_3}^i g^{k_1} g^{k_2} g^{k_3} + \dots \right] dg^j X_i \quad , \quad (2.4)$$

$$\omega^i(g) = \left[\delta_j^i + \frac{1}{2!}c_{jk}^i g^k + \sum_{n=2}^{\infty} \frac{1}{(n+1)!} c_{jk_1}^{h_1} c_{h_1 k_2}^{h_2} \dots c_{h_{n-1} k_{n-1}}^{h_{n-1}} c_{h_{n-1} k_n}^i g^{k_1} g^{k_2} \dots g^{k_{n-1}} g^{k_n} \right] dg^j \quad . \quad (2.5)$$

²Calligraphic \mathcal{G} , \mathcal{L} , \mathcal{W} will denote both the Lie algebras and their underlying vector spaces; V , W etc. will be used for vector spaces that are not necessarily Lie algebras.

Looking at (2.5), it is evident that the redefinition

$$g^l \rightarrow \lambda g^l \quad (2.6)$$

of *some* coordinates g^l will produce an expansion of the MC one-forms $\omega^i(g, \lambda)$ as a sum of one-forms $\omega^{i,\alpha}(g)$ on G multiplied by the corresponding powers λ^α of λ .

2.1 The Lie algebras $\mathcal{G}(N)$ generated from $\mathcal{G} = V_0 \oplus V_1$

Consider, as a first example, the splitting of \mathcal{G}^* into the sum of two (arbitrary) vector subspaces,

$$\mathcal{G}^* = V_0^* \oplus V_1^*, \quad (2.7)$$

V_0^*, V_1^* being generated by the MC forms $\omega^{i_0}(g), \omega^{i_1}(g)$ of \mathcal{G}^* with indexes corresponding, respectively, to the unmodified and modified parameters,

$$g^{i_0} \rightarrow g^{i_0}, \quad g^{i_1} \rightarrow \lambda g^{i_1}, \quad i_0 (i_1) = 1, \dots, \dim V_0 (\dim V_1). \quad (2.8)$$

In general, the series of $\omega^{i_0}(g, \lambda) \in V_0^*, \omega^{i_1}(g, \lambda) \in V_1^*$, will involve all powers of λ ,

$$\omega^{i_p}(g, \lambda) = \sum_{\alpha=0}^{\infty} \lambda^\alpha \omega^{i_p, \alpha}(g) = \omega^{i_p, 0}(g) + \lambda \omega^{i_p, 1}(g) + \lambda^2 \omega^{i_p, 2}(g) + \dots; \quad p = 0, 1, \quad (2.9)$$

$\omega^{i_p}(g, 1) = \omega^{i_p}(g)$. We will see in the following sections what restrictions on \mathcal{G} make zero certain coefficient one-forms $\omega^{i_p, \alpha}$.

With the above notation, the MC equations (2.1) for \mathcal{G} can be rewritten as

$$d\omega^{k_s} = -\frac{1}{2} c_{i_p j_q}^{k_s} \omega^{i_p} \wedge \omega^{j_q} \quad (p, q, s = 0, 1) \quad (2.10)$$

or, explicitly

$$d\omega^{k_0} = -\frac{1}{2} c_{i_0 j_0}^{k_0} \omega^{i_0} \wedge \omega^{j_0} - c_{i_0 j_1}^{k_0} \omega^{i_0} \wedge \omega^{j_1} - \frac{1}{2} c_{i_1 j_1}^{k_0} \omega^{i_1} \wedge \omega^{j_1} \quad (2.11)$$

$$d\omega^{k_1} = -\frac{1}{2} c_{i_0 j_0}^{k_1} \omega^{i_0} \wedge \omega^{j_0} - c_{i_0 j_1}^{k_1} \omega^{i_0} \wedge \omega^{j_1} - \frac{1}{2} c_{i_1 j_1}^{k_1} \omega^{i_1} \wedge \omega^{j_1}. \quad (2.12)$$

Inserting now the expansions (2.9) into the MC equations (2.10) and using (A.1) in the Appendix, the MC equations are expanded in powers of λ :

$$\sum_{\alpha=0}^{\infty} \lambda^\alpha d\omega^{k_s, \alpha} = \sum_{\alpha=0}^{\infty} \lambda^\alpha \left[-\frac{1}{2} c_{i_p j_q}^{k_s} \sum_{\beta=0}^{\alpha} \omega^{i_p, \beta} \wedge \omega^{j_q, \alpha-\beta} \right]. \quad (2.13)$$

The equality of the two λ -polynomials in (2.13) requires the equality of the coefficients of equal power λ^α . This implies that the coefficient one-forms $\omega^{i_p, \alpha}$ in the expansions (2.9) satisfy the identities:

$$d\omega^{k_s, \alpha} = -\frac{1}{2} c_{i_p j_q}^{k_s} \sum_{\beta=0}^{\alpha} \omega^{i_p, \beta} \wedge \omega^{j_q, \alpha-\beta} \quad (p, q, s = 0, 1) \quad (2.14)$$

We can rewrite (2.14) in the form

$$d\omega^{k_s, \alpha} = -\frac{1}{2} C_{i_p, \beta \ j_q, \gamma}^{k_s, \alpha} \omega^{i_p, \beta} \wedge \omega^{j_q, \gamma}, \quad C_{i_p, \beta \ j_q, \gamma}^{k_s, \alpha} = \begin{cases} 0, & \text{if } \beta + \gamma \neq \alpha \\ c_{i_p j_q}^{k_s}, & \text{if } \beta + \gamma = \alpha \end{cases}. \quad (2.15)$$

We now ask ourselves whether we can use the expansion coefficients $\omega^{k_0, \alpha}$, $\omega^{k_1, \beta}$ up to given orders $N_0 \geq 0$, $N_1 \geq 0$, $\alpha = 0, 1, \dots, N_0$, $\beta = 0, 1, \dots, N_1$, so that eq. (2.15) (or (2.14)) determines the MC equations of a new Lie algebra. The answer is given by the following

Theorem 1 Let \mathcal{G} be a Lie algebra, and $\mathcal{G} = V_0 \oplus V_1$ (no subalgebra condition is assumed neither for V_0 or V_1). Let $\{\omega^i\}$, $\{\omega^{i_0}\}$, $\{\omega^{i_1}\}$ ($i = 1, \dots, \dim \mathcal{G}$, $i_0 = 1, \dots, \dim V_0$, $i_1 = 1, \dots, \dim V_1$) be, respectively, the bases of the \mathcal{G}^* , V_0^* and V_1^* dual vector spaces. Then, the vector space generated by

$$\{\omega^{i_0, 0}, \omega^{i_0, 1}, \dots, \omega^{i_0, N_0}; \omega^{i_1, 0}, \omega^{i_1, 1}, \dots, \omega^{i_1, N_1}\}, \quad (2.16)$$

together with the MC eqs. (2.15) for the structure constants

$$C_{i_p, \beta \ j_q, \gamma}^{k_s, \alpha} = \begin{cases} 0, & \text{if } \beta + \gamma \neq \alpha \\ c_{i_p j_q}^{k_s}, & \text{if } \beta + \gamma = \alpha \end{cases} \quad (\alpha, \beta, \gamma = 1, \dots, N; p, q, s = 0, 1), \quad (2.17)$$

determines a Lie algebra $\mathcal{G}(N)$ for each expansion order $N \geq 0$ of dimension $\dim \mathcal{G}(N) = (N + 1) \dim \mathcal{G}$.

Proof.

Consider the one-forms

$$\{\omega^{i_0, \alpha_0}; \omega^{i_1, \alpha_1}\} = \{\omega^{i_0, 0}, \omega^{i_0, 1}, \dots, \omega^{i_0, N_0}; \omega^{i_1, 0}, \omega^{i_1, 1}, \dots, \omega^{i_1, N_1}\} \quad (2.18)$$

where we have not assumed *a priori* the same range for the expansions of the one-forms of V_0^* and V_1^* . To see whether the vector space $V^*(N_0, N_1)$ of basis (2.18) determines a Lie algebra $\mathcal{G}(N_0, N_1)$, it is sufficient to check that a) that the exterior algebra generated by (2.18) is closed³ under the exterior derivative d and that a) the Jacobi identities (JI) for \mathcal{G} are satisfied.

To have closure under d we need that the *r.h.s.* of eqs. (2.15) does not contain one-forms that are not already present in (2.18). Consider the forms ω^{i_s, β_s} , $s = 0, 1$ that contribute to $d\omega^{k_s, \alpha_s}$ up to order $\alpha = N_s$. Looking at eqs. (2.14) it follows trivially that

$$N_0 = N_1 \quad (= N). \quad (2.19)$$

To check the JI for $\mathcal{G}(N)$, it is sufficient to see that $dd\omega^{k_s, \alpha} \equiv 0$ in (2.15) is consistent with the definition of $C_{i_p, \beta \ j_q, \gamma}^{k_s, \alpha}$. Eq. (2.15) gives

$$0 = C_{i_p, \beta \ j_q, \gamma}^{k_s, \alpha} C_{l_t, \rho \ m_u, \sigma}^{i_p, \beta} \omega^{j_q, \gamma} \wedge \omega^{l_t, \rho} \wedge \omega^{m_u, \sigma} \quad (\alpha, \beta, \gamma, \rho, \sigma = 1, \dots, N), \quad (2.20)$$

³An algebra of forms closed under d defines in general a free differential algebra (FDA) [43, 36, 44, 27] (FDA's were called Cartan integrable systems in [36]). When the generating forms are one-forms, the FDA corresponds to a Lie algebra.

which implies

$$C_{i_p, \beta}^{k_s, \alpha} C_{l_t, \rho m_u, \sigma}^{i_p, \beta} = 0 . \quad (2.21)$$

Now, on account of definition (2.17), the terms in the *l.h.s.* above are either zero (when $\alpha \neq \gamma + \rho + \sigma$) or give zero due to the JI for \mathcal{G} , $c_{i_p[j_q]c_{l_t m_u}}^{k_s} = 0$. Thus, the $C_{i_p, \beta}^{k_s, \alpha}$ satisfy the JI (2.21) and define the Lie algebra $\mathcal{G}(N, N) \equiv \mathcal{G}(N)$, *q.e.d.*

Explicitly, the resulting algebras for the first orders are:

$N = 0$, $\mathcal{G}(0)$:

$$d\omega^{k_s, 0} = -\frac{1}{2}c_{i_p j_q}^{k_s} \omega^{i_p, 0} \wedge \omega^{j_q, 0} \quad (p, q, s = 0, 1) , \quad (2.22)$$

i.e., $\mathcal{G}(0)$ reproduces the original algebra \mathcal{G} .

$N = 1$, $\mathcal{G}(1)$:

$$d\omega^{k_s, 0} = -\frac{1}{2}c_{i_p j_q}^{k_s} \omega^{i_p, 0} \wedge \omega^{j_q, 0} , \quad (2.23)$$

$$d\omega^{k_s, 1} = -c_{i_p j_q}^{k_s} \omega^{i_p, 0} \wedge \omega^{j_q, 1} \quad (p, q, s = 0, 1) \quad (2.24)$$

$N = 2$, $\mathcal{G}(2)$:

$$d\omega^{k_s, 0} = -\frac{1}{2}c_{i_p j_q}^{k_s} \omega^{i_p, 0} \wedge \omega^{j_q, 0} , \quad (2.25)$$

$$d\omega^{k_s, 1} = -c_{i_p j_q}^{k_s} \omega^{i_p, 0} \wedge \omega^{j_q, 1} , \quad (2.26)$$

$$d\omega^{k_s, 2} = -c_{i_p j_q}^{k_s} \omega^{i_p, 0} \wedge \omega^{j_q, 2} - \frac{1}{2}c_{i_p j_q}^{k_s} \omega^{i_p, 1} \wedge \omega^{j_q, 1} \quad (p, q, s = 0, 1) . \quad (2.27)$$

Remark. Since $\omega^{i_p, 0}(g) \neq \omega^{i_p}(g)$, one might wonder *e.g.* how the MC eqs. for $\mathcal{G}(0) = \mathcal{G}$ can be satisfied by $\omega^{i_p, 0}(g)$. The $\dim \mathcal{G}$ MC forms $\omega^{i_p}(g)$ are LI forms on the group manifold G of \mathcal{G} . The $(N+1)\dim \mathcal{G}$ $\omega^{i_p, \alpha}(g)$ ($\alpha = 0, 1, \dots, N$) determined by the expansions (2.9) are also one-forms on G , but they are no longer LI under G -translations. They cannot be, since there are only $\dim G = r$ linearly independent MC forms on G . Nevertheless, eqs. (2.15) determine the MC relations that will be satisfied by the MC forms on the manifold of the *higher dimensional* group $G(N)$ associated with $\mathcal{G}(N)$. These MC forms on $G(N)$ will depend on the $(N+1)\dim \mathcal{G}(N)$ coordinates of $G(N)$ associated with the generators (forms) $X_{i_p, \alpha}$ ($\omega^{i_p, \alpha}$) that determine $\mathcal{G}(N)$ ($\mathcal{G}^*(N)$).

2.2 Structure of the Lie algebras $\mathcal{G}(N)$

Let $V_{p, \alpha}$ be, at each order $\alpha = 0, 1, \dots, N$, the vector space spanned by the generators $X_{i_p, \alpha}$, $p = 0, 1$; clearly, $V_{p, \alpha} \approx V_p$. Let

$$W_\alpha = V_{0, \alpha} \oplus V_{1, \alpha} \quad , \quad \mathcal{G}(N) = \bigoplus_{\alpha=0}^N W_\alpha \quad . \quad (2.28)$$

We first notice that $\mathcal{G}(N-1)$ is a vector subspace of $\mathcal{G}(N)$, but not a subalgebra for $N \geq 2$. Indeed, for $N \geq 2$ there always exist $\alpha, \beta \leq N-1$ such that $\alpha + \beta = N$. Denoting by $C_{i_p, \alpha j_q, \beta}^{(N) k_s, \gamma}$ and $C_{i_p, \alpha j_q, \beta}^{(N-1) k_s, \gamma}$ the structure constants of $\mathcal{G}(N)$ and $\mathcal{G}(N-1)$ respectively, one sees that, for $\alpha + \beta = N$, $C_{i_p, \alpha j_q, \beta}^{(N-1) k_s, \alpha + \beta} = 0$ in $\mathcal{G}(N-1)$ (since $\alpha + \beta > N-1$) while, in general, $C_{i_p, \alpha j_q, \beta}^{(N) k_s, \alpha + \beta} \neq 0$ in $\mathcal{G}(N)$. In other words, $\mathcal{G}(N-1)$ is not a subalgebra of $\mathcal{G}(N)$ because the structure constants for the elements of the various subspaces $V_{p, \alpha}$ depend on N and they are different, in general, for $\mathcal{G}(N-1)$ and $\mathcal{G}(N)$. Likewise, $\mathcal{G}(M)$ for $1 \leq M < N$ is not a subalgebra of $\mathcal{G}(N)$.

We show in this section that the Lie algebras $\mathcal{G}(N)$ have a Lie algebra extension structure for $N \geq 1$.

Proposition 1 The Lie algebra $\mathcal{G}(0)$ is a subalgebra of $\mathcal{G}(N)$, for all $N \geq 0$. For $N \geq 1$, \mathcal{W}_N is an abelian ideal $\mathcal{W}_N \subset \mathcal{G}(N)$ and $\mathcal{G}(N)/\mathcal{W}_N = \mathcal{G}(N-1)$ i.e., $\mathcal{G}(N)$ is an extension of $\mathcal{G}(N-1)$ by \mathcal{W}_N which is not semidirect for $N \geq 2$.

Proof.

$\mathcal{G}(0) \subset \mathcal{G}(N)$ is a subalgebra by construction, since $C_{i_p, 0 j_q, 0}^{(N) k_s, \alpha} = 0$, $\alpha = 1, \dots, N$, by eq. (2.15).

For the second part, notice that, since $\alpha + N > N$ for $\alpha \neq 0$, $[W_\alpha, \mathcal{W}_N] = 0$; in particular, \mathcal{W}_N is an abelian subalgebra. Furthermore $[\mathcal{W}_0, \mathcal{W}_N] \subset \mathcal{W}_N$, so that \mathcal{W}_N is an ideal of $\mathcal{G}(N)$. Now, the vector space $\mathcal{G}(N)/\mathcal{W}_N$ is isomorphic to $\mathcal{G}(N-1)$. $\mathcal{G}(N-1)$ is a Lie algebra the MC equations of which are (2.15), and $\mathcal{G}(N)/\mathcal{W}_N \approx \mathcal{G}(N-1)$. Since $\mathcal{G}(N-1)$ is not a subalgebra of $\mathcal{G}(N)$ for $N \geq 2$, the extension is not semidirect, *q.e.d.*

2.3 The limiting cases $V_0 = 0, V_1 = V$ and $V_0 = V, V_1 = 0$

When $V_1 = V$, all the group parameters are modified by (2.8). In this case $\mathcal{G}(0)$ is the trivial $\mathcal{G}(0) = 0$ subalgebra of $\mathcal{G}(N)$. The first order $N = 1$, $\omega^{i_1, 1} = dg^{i_1}$, corresponds to an abelian algebra with the same dimension as \mathcal{G} (in fact, $\mathcal{G}(1)$ is the IW contraction of \mathcal{G} with respect to the trivial $V_0 = 0$ subalgebra). For $N \geq 2$ we will have extensions with the structure in Prop. 1.

For the other limiting case, $V_1 = 0$, there is obviously no expansion and we have $\mathcal{G}(0) = \mathcal{G}$.

3 The case in which V_0 is a subalgebra $\mathcal{L}_0 \subset \mathcal{G}$

Let $\mathcal{G} = V_0 \oplus V_1$ as before, where now V_0 is a subalgebra \mathcal{L}_0 of \mathcal{G} . Then,

$$c_{i_0 j_0}^{k_1} = 0 \quad (i_p = 1, \dots, \dim V_p, p = 0, 1) \quad , \quad (3.1)$$

and the basis one-forms ω^{i_0} are associated with the (sub)group parameters g^{i_0} unmodified under the rescaling (2.8). The MC equations for \mathcal{G} become

$$d\omega^{k_0} = -\frac{1}{2}c_{i_0 j_0}^{k_0} \omega^{i_0} \wedge \omega^{j_0} - c_{i_0 j_1}^{k_0} \omega^{i_0} \wedge \omega^{j_1} - \frac{1}{2}c_{i_1 j_1}^{k_0} \omega^{i_1} \wedge \omega^{j_1} \quad , \quad (3.2)$$

$$d\omega^{k_1} = -c_{i_0 j_1}^{k_1} \omega^{i_0} \wedge \omega^{j_1} - \frac{1}{2}c_{i_1 j_1}^{k_1} \omega^{i_1} \wedge \omega^{j_1} \quad . \quad (3.3)$$

Using (3.1) in eq. (2.5), one finds that the expansions of $\omega^{i_0}(g, \lambda)$ ($\omega^{i_1}(g, \lambda)$) start with the power λ^0 (λ^1):

$$\omega^{i_0}(g, \lambda) = \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \omega^{i_0, \alpha}(g) = \omega^{i_0, 0}(g) + \lambda \omega^{i_0, 1}(g) + \lambda^2 \omega^{i_0, 2}(g) + \dots \quad (3.4)$$

$$\omega^{i_1}(g, \lambda) = \sum_{\alpha=1}^{\infty} \lambda^{\alpha} \omega^{i_1, \alpha}(g) = \lambda \omega^{i_1, 1}(g) + \lambda^2 \omega^{i_1, 2}(g) + \lambda^3 \omega^{i_1, 3}(g) + \dots \quad (3.5)$$

Inserting them into the MC equations (3.2) and (3.3) and using eq. (A.1) when the double sums begin with (0, 0), (0, 1) and (1, 1), we get

$$\begin{aligned} \sum_{\alpha=0}^{\infty} \lambda^{\alpha} d\omega^{k_0, \alpha} = & -\frac{1}{2} c_{i_0 j_0}^{k_0} \omega^{i_0, 0} \wedge \omega^{j_0, 0} + \lambda \left[-c_{i_0 j_0}^{k_0} \omega^{i_0, 0} \wedge \omega^{j_0, 1} - c_{i_0 j_1}^{k_0} \omega^{i_0, 0} \wedge \omega^{j_1, 1} \right] + \\ & + \sum_{\alpha=2}^{\infty} \lambda^{\alpha} \left[-\frac{1}{2} c_{i_0 j_0}^{k_0} \sum_{\beta=0}^{\alpha} \omega^{i_0, \beta} \wedge \omega^{j_0, \alpha-\beta} - c_{i_0 j_1}^{k_0} \sum_{\beta=0}^{\alpha-1} \omega^{i_0, \beta} \wedge \omega^{j_1, \alpha-\beta} - \right. \\ & \left. - \frac{1}{2} c_{i_1 j_1}^{k_0} \sum_{\beta=1}^{\alpha-1} \omega^{i_1, \beta} \wedge \omega^{j_1, \alpha-\beta} \right] , \end{aligned} \quad (3.6)$$

$$\begin{aligned} \sum_{\alpha=1}^{\infty} \lambda^{\alpha} d\omega^{k_1, \alpha} = & -\lambda c_{i_0 j_1}^{k_1} \omega^{i_0, 0} \wedge \omega^{j_1, 1} + \\ & + \sum_{\alpha=2}^{\infty} \lambda^{\alpha} \left[-c_{i_0 j_1}^{k_1} \sum_{\beta=0}^{\alpha-1} \omega^{i_0, \beta} \wedge \omega^{j_1, \alpha-\beta} - \frac{1}{2} c_{i_1 j_1}^{k_1} \sum_{\beta=1}^{\alpha-1} \omega^{i_1, \beta} \wedge \omega^{j_1, \alpha-\beta} \right] . \end{aligned} \quad (3.7)$$

Again, the equality of the coefficients of equal power λ^{α} in (3.6), (3.7) leads to the equalities:
 $\alpha = 0$:

$$d\omega^{k_0, 0} = -\frac{1}{2} c_{i_0 j_0}^{k_0} \omega^{i_0, 0} \wedge \omega^{j_0, 0} ; \quad (3.8)$$

$\alpha = 1$:

$$d\omega^{k_0, 1} = -c_{i_0 j_0}^{k_0} \omega^{i_0, 0} \wedge \omega^{j_0, 1} - c_{i_0 j_1}^{k_0} \omega^{i_0, 0} \wedge \omega^{j_1, 1} , \quad (3.9)$$

$$d\omega^{k_1, 1} = -c_{i_0 j_1}^{k_1} \omega^{i_0, 0} \wedge \omega^{j_1, 1} ; \quad (3.10)$$

$\alpha \geq 2$:

$$d\omega^{k_0, \alpha} = -\frac{1}{2} c_{i_0 j_0}^{k_0} \sum_{\beta=0}^{\alpha} \omega^{i_0, \beta} \wedge \omega^{j_0, \alpha-\beta} - c_{i_0 j_1}^{k_0} \sum_{\beta=0}^{\alpha-1} \omega^{i_0, \beta} \wedge \omega^{j_1, \alpha-\beta} - \frac{1}{2} c_{i_1 j_1}^{k_0} \sum_{\beta=1}^{\alpha-1} \omega^{i_1, \beta} \wedge \omega^{j_1, \alpha-\beta} , \quad (3.11)$$

$$d\omega^{k_1, \alpha} = -c_{i_0 j_1}^{k_1} \sum_{\beta=0}^{\alpha-1} \omega^{i_0, \beta} \wedge \omega^{j_1, \alpha-\beta} - \frac{1}{2} c_{i_1 j_1}^{k_1} \sum_{\beta=1}^{\alpha-1} \omega^{i_1, \beta} \wedge \omega^{j_1, \alpha-\beta} . \quad (3.12)$$

To allow for a different range in the orders α of each $\omega^{i_p, \alpha}$, we now denote the coefficient one-forms in (3.4) ((3.5)) ω^{i_0, α_0} (ω^{i_1, α_1}), $\alpha_0 = 0, 1, \dots, N_0$ ($\alpha_1 = 1, 2, \dots, N_1$). With this notation, the above relations take the generic form

$$d\omega^{k_s, \alpha_s} = -\frac{1}{2} C_{i_p, \beta_p, j_q, \gamma_q}^{k_s, \alpha_s} \omega^{i_p, \beta_p} \wedge \omega^{j_q, \gamma_q} \quad , \quad (3.13)$$

where

$$C_{i_p, \alpha_p, j_q, \alpha_q}^{k_s, \alpha_s} = \begin{cases} 0, & \text{if } \beta_p + \gamma_q \neq \alpha_s \\ c_{i_p, j_q}^{k_s}, & \text{if } \beta_p + \gamma_q = \alpha_s \end{cases} \quad \begin{matrix} p, q, s = 0, 1 \\ i_{p,q,s} = 1, 2, \dots, \dim V_{p,q,s} \\ \alpha_0, \beta_0, \gamma_0 = 0, 1, \dots, N_0 \\ \alpha_1, \beta_1, \gamma_1 = 1, 2, \dots, N_1 \end{matrix} \quad (3.14)$$

As in the preceding case, we now ask ourselves whether the expansion coefficients ω^{k_0, α_0} , ω^{k_1, α_1} up to a given order N_0, N_1 determine the MC equations (3.13) of a new Lie algebra $\mathcal{G}(N_0, N_1)$. It is obvious from (3.8) that the zero order of the expansion in λ corresponds to $N_0 = 0 = N_1$ (omitting all ω^{i_1, α_1} and thus allowing N_1 to be zero), and that $\mathcal{G}(0, 0) = \mathcal{L}_0$. It is seen directly that the terms up to first order give two possibilities: $\mathcal{G}(0, 1)$ (eqs. (3.8), (3.10) for $\omega^{k_0, 0}, \omega^{k_1, 1}$) and $\mathcal{G}(1, 1)$ (eqs. (3.8), (3.9), (3.10) for $\omega^{k_0, 0}, \omega^{k_1, 1}, \omega^{k_0, 1}$). Thus, we see that now (and due to (3.1)) one does not need to retain *all* ω^{i_p, α_p} up to a given order to obtain a Lie algebra. To look at the general $N_0 \geq 0, N_1 \geq 1$ case, consider the vector space $V^*(N_0, N_1)$, generated by

$$\{\omega^{i_0, \alpha_0}; \omega^{i_1, \alpha_1}\} = \{\omega^{i_0, 0}, \omega^{i_0, 1}, \omega^{i_0, 2}, \dots, \omega^{i_0, N_0}; \omega^{i_1, 1}, \omega^{i_1, 2}, \dots, \omega^{i_1, N_1}\} \quad . \quad (3.15)$$

To see that it determines a Lie algebra $\mathcal{G}(N_0, N_1)$ of dimension

$$\dim \mathcal{G}(N_0, N_1) = (N_0 + 1) \dim V_0 + N_1 \dim V_1 \quad , \quad (3.16)$$

we first notice that the JI in $\mathcal{G}(N_0, N_1)$ will follow from the JI in \mathcal{G} . To find the conditions that N_0 and N_1 must satisfy to have closure under d , we look at the orders β_p of the forms ω^{i_p, β_p} that appear in the expression (3.13) of $d\omega^{k_s, \alpha_s}$ up to a given order $\alpha_s \geq s$. Looking at eqs. (3.8) to (3.12) we find the following table:

| $\alpha_s \geq s$ | ω^{i_0, β_0} | ω^{i_1, β_1} |
|---------------------------|-----------------------------|-------------------------|
| $d\omega^{k_0, \alpha_0}$ | $\beta_0 \leq \alpha_0$ | $\beta_1 \leq \alpha_0$ |
| $d\omega^{k_1, \alpha_1}$ | $\beta_0 \leq \alpha_1 - 1$ | $\beta_1 \leq \alpha_1$ |

Orders β_p of the forms ω^{i_p, β_p} that contribute to $d\omega^{k_s, \alpha_s}$

Since there must be enough one-forms in (3.15) for the MC equations (3.13) to be satisfied, the $N_0 + 1$ and N_1 one-forms ω^{i_0, α_0} ($\alpha_0 = 0, 1, \dots, N_0$) and ω^{i_1, α_1} ($\alpha_1 = 1, 2, \dots, N_1$) in (3.15) should include, at least, those appearing in their differentials. Thus, the previous table implies the reverse inequalities

| $\alpha_s \geq s$ | ω^{i_0, β_0} | ω^{i_1, β_1} |
|---------------------------|-------------------------|-------------------------|
| $d\omega^{k_0, \alpha_0}$ | $N_0 \geq N_0$ | $N_1 \geq N_0$ |
| $d\omega^{k_1, \alpha_1}$ | $N_0 \geq N_1 - 1$ | $N_1 \geq N_1$ |

Conditions on the number N_0 (N_1) of one-forms ω^{i_0, α_0} (ω^{i_1, α_1})

Hence, in this case there are two ways of cutting the expansions (3.4), (3.5), namely for

$$N_1 = N_0 , \quad (3.17)$$

$$\text{or} \quad N_1 = N_0 + 1 . \quad (3.18)$$

Besides (2.19) there is now an additional type of solutions, eq. (3.18). For the $N_0 = 0, N_1 = 1$ values eq. (3.16) yields $\dim \mathcal{G}(0, 1) = \dim \mathcal{G}$. Then, $\alpha_0 = 0$ and $\alpha_1 = 1$ only, the label α_p may be dropped and the structure constants (3.14) for $\mathcal{G}(0, 1)$ read

$$C_{i_p j_q}^{k_s} = \begin{cases} 0, & \text{if } p + q \neq s \\ c_{i_p j_q}^{k_s}, & \text{if } p + q = s \end{cases} \quad \begin{matrix} p = 0, 1 \\ i_{p,q,s} = 1, 2, \dots, \dim V_{p,q,s} \end{matrix} , \quad (3.19)$$

which shows that V_1 is an abelian ideal of $\mathcal{G}(0, 1)$. Hence, $\mathcal{G}(0, 1)$ is just the (simple) IW contraction of \mathcal{G} with respect to the subalgebra \mathcal{L}_0 , as it may be seen by taking the $\lambda \rightarrow 0$ limit in (3.6)-(3.7), which reduce to eqs. (3.8) and (3.10). We can thus state the following

Theorem 2 Let $\mathcal{G} = V_0 \oplus V_1$, where V_0 is a subalgebra \mathcal{L}_0 . Let the coordinates g^{i_p} of G be rescaled by $g^{i_0} \rightarrow g^{i_0}, g^{i_1} \rightarrow \lambda g^{i_1}$ (eq. (2.8)). Then, the coefficient one-forms $\{\omega^{i_0, \alpha_0}; \omega^{i_1, \alpha_1}\}$ of the expansions (3.4), (3.5) of the Maurer-Cartan forms of \mathcal{G}^* determine Lie algebras $\mathcal{G}(N_0, N_1)$ when $N_1 = N_0$ or $N_1 = N_0 + 1$ of dimension $\dim \mathcal{G}(N_0, N_1) = (N_0 + 1) \dim V_0 + N_1 \dim V_1$ and with structure constants (3.14),

$$C_{i_p, \beta_p j_q, \gamma_q}^{k_s, \alpha_s} = \begin{cases} 0, & \text{if } \beta_p + \gamma_q \neq \alpha_s \\ c_{i_p j_q}^{k_s}, & \text{if } \beta_p + \gamma_q = \alpha_s \end{cases} \quad \begin{matrix} p, q, s = 0, 1 \\ i_{p,q,s} = 1, 2, \dots, \dim V_{p,q,s} \\ \alpha_0, \beta_0, \gamma_0 = 0, 1, \dots, N_0 \\ \alpha_1, \beta_1, \gamma_1 = 1, 2, \dots, N_1 \end{matrix}$$

In particular, $\mathcal{G}(0, 0) = \mathcal{L}_0$ and $\mathcal{G}(0, 1)$ (eq. (3.18) for $N_0 = 0$) is the simple IW contraction of \mathcal{G} with respect to the subalgebra \mathcal{L}_0 .

3.1 The case in which V_1 is a symmetric coset

Let us now particularize to the case in which $\mathcal{G}/\mathcal{L}_0 = V_1$ is a symmetric coset *i.e.*,

$$[V_0, V_0] \subset V_0, \quad [V_0, V_1] \subset V_1, \quad [V_1, V_1] \subset V_0, \quad (3.20)$$

($[V_p, V_q] \subset V_{p+q}, (p+q) \bmod 2$). This applies, for instance, to all superalgebras where V_0 is the bosonic subspace and V_1 the fermionic one. Then, if $c_{i_p j_q}^{k_s}$ ($p, q, s = 0, 1; i_p = 1, \dots, \dim V_p$) are the structure constants of \mathcal{G} , $c_{i_p j_q}^{k_s} = 0$ if $s \neq (p+q) \bmod 2$, the MC equations reduce to

$$d\omega^{k_0} = -\frac{1}{2} c_{i_0 j_0}^{k_0} \omega^{i_0} \wedge \omega^{j_0} - \frac{1}{2} c_{i_1 j_1}^{k_0} \omega^{i_1} \wedge \omega^{j_1} \quad (3.21)$$

$$d\omega^{k_1} = -c_{i_0 j_1}^{k_1} \omega^{i_0} \wedge \omega^{j_1}, \quad (3.22)$$

and one can state the following

Proposition 2 Let G and \mathcal{G} be as in Th. 2, and let further V_1 be a symmetric space, eq. (3.20). Then, the rescaling (2.8) leads to an even (odd) power series in λ for the MC forms $\omega^{i_0}(g, \lambda)$ ($\omega^{i_1}(g, \lambda)$):

$$\begin{aligned}\omega^{i_0}(g, \lambda) &= \omega^{i_0,0}(g) + \lambda^2 \omega^{i_0,2}(g) + \lambda^4 \omega^{i_0,4}(g) + \dots \\ \omega^{i_1}(g, \lambda) &= \lambda \omega^{i_1,1}(g) + \lambda^3 \omega^{i_1,3}(g) + \lambda^5 \omega^{i_1,5}(g) + \dots ,\end{aligned}\quad (3.23)$$

namely, $\omega^{i_{\bar{\alpha}}}(g, \lambda) = \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \omega^{i_{\bar{\alpha}},\alpha}(g)$; $\bar{\alpha} = \alpha \pmod{2}$.

Proof.

Under (2.8) $dg^{i_0} \rightarrow dg^{i_0}$, $dg^{i_1} \rightarrow \lambda dg^{i_1}$, which contributes with λ^0 (λ) to $\omega^{i_0}(g, \lambda)$ ($\omega^{i_1}(g, \lambda)$); $c_{j_q k_s}^{i_p}$ vanish trivially unless $p = (q + s) \pmod{2}$. Then, under (2.8), the $g^{k_s} dg^{j_q}$ terms in (2.5) with one g^{k_s} rescale as

$$\begin{aligned}p = 0 & : c_{j_0 k_0}^{i_0} g^{k_0} dg^{j_0} \rightarrow c_{j_0 k_0}^{i_0} g^{k_0} dg^{j_0} , \quad c_{j_1 k_1}^{i_0} g^{k_1} dg^{j_1} \rightarrow \lambda^2 c_{j_1 k_1}^{i_0} g^{k_1} dg^{j_1} ; \\ p = 1 & : c_{j_0 k_1}^{i_1} g^{k_1} dg^{j_0} \rightarrow \lambda c_{j_0 k_1}^{i_1} g^{k_1} dg^{j_0} ,\end{aligned}\quad (3.24)$$

so that the powers λ^0 and λ^2 (λ) contribute to $\omega^{i_0}(g, \lambda)$ ($\omega^{i_1}(g, \lambda)$). For the terms in (2.5) involving the products of n g^{k_s} 's,

$$c_{j_q k_{s_1}}^{h_{t_1}} c_{h_{t_1} k_{s_2}}^{h_{t_2}} \dots c_{h_{t_{n-2}} k_{t_{n-1}}}^{h_{t_{n-1}}} c_{h_{t_{n-1}} k_{s_n}}^{i_p} g^{k_{s_1}} g^{k_{s_2}} \dots g^{k_{s_{n-1}}} g^{k_{s_n}} dg^{j_q} , \quad (3.25)$$

the fact that $V_1 = \mathcal{G}/\mathcal{L}_0$ is a symmetric space requires that $p = q + s_1 + s_2 + \dots + s_n \pmod{2}$. Thus, after the rescaling (2.8), only even (odd) powers of λ , from λ^0 (λ) up to the closest (lower or equal to) $n + 1$ even (odd) power λ^{n+1} , contribute to $\omega^{i_0}(g, \lambda)$ ($\omega^{i_1}(g, \lambda)$), *q.e.d.*

3.1.1 Structure of $\mathcal{G}(N_0, N_1)$ in the symmetric coset case

Inserting the power series above into the MC equations (3.21) and (3.22), we arrive at the equalities:

$$d\omega^{k_0, 2\sigma} = -\frac{1}{2} c_{i_0 j_0}^{k_0} \sum_{\rho=0}^{\sigma} \omega^{i_0, 2\rho} \wedge \omega^{j_0, 2(\sigma-\rho)} - \frac{1}{2} c_{i_1 j_1}^{k_0} \sum_{\rho=1}^{\sigma} \omega^{i_1, 2\rho-1} \wedge \omega^{j_1, 2(\sigma-\rho)+1} , \quad (3.26)$$

$$d\omega^{k_1, 2\sigma+1} = -c_{i_0 j_1}^{k_1} \sum_{\rho=0}^{\sigma} \omega^{i_0, 2\rho} \wedge \omega^{j_1, 2(\sigma-\rho)+1} , \quad (3.27)$$

where the expansion orders α are either $\alpha = 2\sigma$ or $\alpha = 2\sigma + 1$. From them it follows that the vector spaces generated by

$$\{\omega^{i_0,0}, \omega^{i_0,2}, \omega^{i_0,4}, \dots, \omega^{i_0, N_0}; \omega^{i_1,1}, \omega^{i_1,3}, \dots, \omega^{i_1, N_1}\} , \quad (3.28)$$

where $N_0 \geq 0$ (and even) and $N_1 \geq 1$ (and odd), will determine a Lie algebra when

$$N_1 = N_0 - 1 , \quad (3.29)$$

$$\text{or } N_1 = N_0 + 1 . \quad (3.30)$$

Notice that we have a new type of solutions (3.29) with respect to the preceding case (eqs. (3.17), (3.18)), and that the previous solution $N_0 = N_1$ is not allowed now since N_0

(N_1) is necessarily even (odd). Then, for the symmetric case, the algebras $\mathcal{G}(N_0, N_1)$ may also be denoted $\mathcal{G}(N)$, where $N = \max\{N_0, N_1\}$, and are obtained at each order by adding alternatively copies of V_0 and V_1 . Its structure constants are given by

$$C_{i_{\bar{\beta}, \beta} j_{\bar{\gamma}, \gamma}}^{k_{\bar{\alpha}, \alpha}} = \begin{cases} 0, & \text{if } \beta + \gamma \neq \alpha \\ c_{i_{\bar{\beta}, \beta} j_{\bar{\gamma}, \gamma}}^{k_{\bar{\alpha}, \alpha}}, & \text{if } \beta + \gamma = \alpha ; \bar{\alpha} = \alpha \pmod{2}, \bar{\beta} = \beta \pmod{2}, \bar{\gamma} = \gamma \pmod{2} \end{cases} \quad (3.31)$$

Let us write explicitly the MC eqs. for the first algebras obtained. If we allow for $N_1 = 0$, we get the trivial case

$$\mathcal{G}(0, 0) = \mathcal{G}(0):$$

$$d\omega^{k_0, 0} = -\frac{1}{2}c_{i_0 j_0}^{k_0} \omega^{i_0, 0} \wedge \omega^{j_0, 0} \quad (3.32)$$

i.e., $\mathcal{G}(0, 0)$ is the subalgebra \mathcal{L}_0 of the original algebra \mathcal{G} .

$$\mathcal{G}(0, 1) = \mathcal{G}(1):$$

$$d\omega^{k_0, 0} = -\frac{1}{2}c_{i_0 j_0}^{k_0} \omega^{i_0, 0} \wedge \omega^{j_0, 0}, \quad (3.33)$$

$$d\omega^{k_1, 1} = -c_{i_0 j_1}^{k_1} \omega^{i_0, 0} \wedge \omega^{j_1, 1}, \quad (3.34)$$

so that $\mathcal{G}(0, 1)$ is again (Th. 2) the IW contraction of \mathcal{G} with respect to \mathcal{L}_0 .

$$\mathcal{G}(2, 1) = \mathcal{G}(2):$$

$$d\omega^{k_0, 0} = -\frac{1}{2}c_{i_0 j_0}^{k_0} \omega^{i_0, 0} \wedge \omega^{j_0, 0}, \quad (3.35)$$

$$d\omega^{k_1, 1} = -c_{i_0 j_1}^{k_1} \omega^{i_0, 0} \wedge \omega^{j_1, 1}, \quad (3.36)$$

$$d\omega^{k_0, 2} = -c_{i_0 j_0}^{k_0} \omega^{i_0, 0} \wedge \omega^{j_0, 2} - \frac{1}{2}c_{i_1 j_1}^{k_0} \omega^{i_1, 1} \wedge \omega^{j_1, 1}. \quad (3.37)$$

The structure of the Lie algebras $\mathcal{G}(N)$ is given by the following

Proposition 3 The Lie algebra $\mathcal{G}(0) = \mathcal{L}_0$ is a subalgebra of $\mathcal{G}(N)$ for all $N \geq 0$. W_α in (2.28) reduces here to

$$W_\alpha = \begin{cases} V_{0, \alpha}, & \text{if } \alpha \text{ even} \\ V_{1, \alpha}, & \text{if } \alpha \text{ odd} \end{cases}. \quad (3.38)$$

For $N \geq 1$, W_N is an abelian ideal \mathcal{W}_N of $\mathcal{G}(N)$ and $\mathcal{G}(N)/\mathcal{W}_N = \mathcal{G}(N-1)$, i.e., $\mathcal{G}(N)$ is an extension of $\mathcal{G}(N-1)$ by \mathcal{W}_N .

Further, for N even and \mathcal{L}_0 abelian, the extension $\mathcal{G}(N)$ of $\mathcal{G}(N-1)$ by \mathcal{W}_N is central.

Proof.

The proof of the first part proceeds as in Prop. 1. For the second part, notice that, for $N \geq 1$, the only thing that prevents the abelian ideal \mathcal{W}_N from being central is its failure to commute with $W_0 \approx \mathcal{L}_0$, since $[W_\alpha, \mathcal{W}_N] = 0$ for $\alpha = 1, 2, \dots, N$. But for N even, $C_{i_0, 0 j_0, N}^{k_0} = c_{i_0 j_0}^{k_0}$, which vanish for \mathcal{L}_0 abelian. Thus \mathcal{W}_N becomes a central ideal, and $\mathcal{G}(N)$ a central extension of $\mathcal{G}(N-1)$ by \mathcal{W}_N , *q.e.d.*

4 Different powers rescaling subordinated to a general splitting of \mathcal{G}

Let us extend now the above results to the case where the group parameters are multiplied by arbitrary integer powers of λ . Let \mathcal{G} be split into a sum of $n + 1$ vector subspaces,

$$\mathcal{G} = V_0 \oplus V_1 \oplus \cdots \oplus V_n = \bigoplus_0^n V_p, \quad (4.1)$$

and let the rescaling

$$g^{i_0} \rightarrow g^{i_0}, \quad g^{i_1} \rightarrow \lambda g^{i_1}, \quad \dots, \quad g^{i_n} \rightarrow \lambda^n g^{i_n} \quad (g^{i_p} \rightarrow \lambda^p g^{i_p}, \quad p = 0, \dots, n) \quad (4.2)$$

of the group coordinates g^{i_p} be subordinated to the splitting (4.1) in an obvious way. We found in the previous section ($p = 0, 1$) that, when the rescaling (2.8) was performed, having V_0 as a subalgebra \mathcal{L}_0 proved to be convenient (though not necessary) since it led to more types of solutions ((3.17)-(3.18), cf. (2.19)). Furthermore, the first order algebra $\mathcal{G}(0, 1)$ for that case was found to be the simple IW contraction of \mathcal{G} with respect to \mathcal{L}_0 . By the same reason, we will consider here conditions on \mathcal{G} that will lead to a richer new algebras structure, including the generalized IW contraction of \mathcal{G} in the sense [11] of Weimar-Woods (W-W)⁴. In terms of the structure constants of \mathcal{G} we will then require

$$c_{i_p j_q}^{k_s} = 0 \quad \text{if } s > p + q \quad (4.3)$$

i.e., that the Lie bracket of elements in V_p, V_q is in $\oplus_s V_s$ for $s \leq p + q$. This condition leads, through (2.5), to a power series expansion of the one-forms ω^{i_p} in V_p^* that, for each $p = 0, 1, \dots, n$, starts precisely with the power λ^p ,

$$\omega^{i_0}(g, \lambda) = \sum_{\alpha=0}^{\infty} \lambda^\alpha \omega^{i_0, \alpha}(g) = \omega^{i_0, 0}(g) + \lambda \omega^{i_0, 1}(g) + \lambda^2 \omega^{i_0, 2}(g) + \dots, \quad (4.4)$$

$$\omega^{i_1}(g, \lambda) = \sum_{\alpha=1}^{\infty} \lambda^\alpha \omega^{i_1, \alpha}(g) = \lambda \omega^{i_1, 1}(g) + \lambda^2 \omega^{i_1, 2}(g) + \lambda^3 \omega^{i_1, 3}(g) + \dots, \quad (4.5)$$

...

$$\omega^{i_n}(g, \lambda) = \sum_{\alpha=n}^{\infty} \lambda^\alpha \omega^{i_n, \alpha}(g) = \lambda^n \omega^{i_n, n}(g) + \lambda^{n+1} \omega^{i_n, n+1}(g) + \dots \quad (4.6)$$

We may extend all the sums so that they begin at $\alpha = 0$ by setting $\omega^{i_p, \alpha} \equiv 0$ when $\alpha < p$. Then, inserting the expansions of $\omega^{i_p, \alpha}$ in the MC eqs. and using (A.1) we get (2.14) for $p, q, s = 0, 1, \dots, n$. If we now introduce the notation ω^{i_p, α_p} with different ranges for the expansion orders, $\alpha_p = p, p + 1, \dots, N_p$ for each p , we see that the MC eqs. take the form

⁴This is, in fact, the most general contraction: any contraction is equivalent to a generalized IW contraction with integer exponents [11]. The generalized IW contraction is defined as follows. Let $\mathcal{G} = \oplus_p V_p$, $p = 0, 1, \dots, n$. Let the basis generators X of \mathcal{G} of each subspace V_p be redefined by $X \rightarrow \lambda^{n_p} X$, where the n_p may be chosen to be integers. Then, it is evident that the generalized IW contraction (the limit $\lambda \rightarrow 0$) exists iff \mathcal{G} is such that $[V_p, V_q] \subset \oplus_s V_s$, where s runs over all the values for which $n_s \leq n_p + n_q$. For (4.1), (4.2) and $n_p \equiv p$, this is equivalent to (4.3) above.

$$d\omega^{k_s, \alpha_s} = -\frac{1}{2} C_{i_p, \beta_p, j_q, \gamma_q}^{k_s, \alpha_s} \omega^{i_p, \beta_p} \wedge \omega^{j_q, \gamma_q} \quad , \quad (4.7)$$

where

$$C_{i_p, \beta_p, j_q, \gamma_q}^{k_s, \alpha_s} = \begin{cases} 0, & \text{if } \beta_p + \gamma_q \neq \alpha_s \\ c_{i_p, j_q}^{k_s}, & \text{if } \beta_p + \gamma_q = \alpha_s \end{cases} \quad \begin{matrix} p, q, s = 0, 1, \dots, n \\ i_{p,q,s} = 1, 2, \dots, \dim V_{p,q,s} \\ \alpha_p, \beta_p, \gamma_p = p, p+1, \dots, N_p \end{matrix} \quad (4.8)$$

and the $c_{i_p, j_q}^{k_s}$ satisfy (4.3). To find now the ω^{i_p, β_p} 's that enter in $d\omega^{k_s, \alpha_s}$, $s = 0, 1, \dots, n$, we need an explicit expression for it. This is found in the Appendix, eqs. (A.7)-(A.10). From them we read that $d\omega^{k_s, \alpha_s}$, $s = 0, 1, \dots, n$, is expressed in terms of products of the forms ω^{i_p, β_p} in the following table:

| $\alpha_s \geq s$ | ω^{i_0, β_0} | ω^{i_1, β_1} | ω^{i_2, β_2} | \dots | ω^{i_n, β_n} |
|---------------------------|-----------------------------|---------------------------------|---------------------------------|----------|-------------------------|
| $d\omega^{k_0, \alpha_0}$ | $\beta_0 \leq \alpha_0$ | $\beta_1 \leq \alpha_0$ | $\beta_2 \leq \alpha_0$ | \dots | $\beta_n \leq \alpha_0$ |
| $d\omega^{k_1, \alpha_1}$ | $\beta_0 \leq \alpha_1 - 1$ | $\beta_1 \leq \alpha_1$ | $\beta_2 \leq \alpha_1$ | \dots | $\beta_n \leq \alpha_1$ |
| $d\omega^{k_2, \alpha_2}$ | $\beta_0 \leq \alpha_2 - 2$ | $\beta_1 \leq \alpha_2 - 1$ | $\beta_2 \leq \alpha_2$ | \dots | $\beta_n \leq \alpha_2$ |
| \vdots | \vdots | \vdots | \vdots | \vdots | \vdots |
| $d\omega^{k_n, \alpha_n}$ | $\beta_0 \leq \alpha_n - n$ | $\beta_1 \leq \alpha_n - n + 1$ | $\beta_2 \leq \alpha_n - n + 2$ | \dots | $\beta_n \leq \alpha_n$ |

Types and orders of the forms ω^{i_p, β_p} needed to express $d\omega^{k_s, \alpha_s}$

Now let $V^*(N_0, \dots, N_n)$ be the vector space generated by

$$\{\omega^{i_0, \alpha_0} ; \omega^{i_1, \alpha_1} ; \dots ; \omega^{i_n, \alpha_n}\} = \{\omega^{i_0, 0}, \omega^{i_0, 1}, \dots, \omega^{i_0, N_0}; \omega^{i_1, 1}, \dots, \omega^{i_1, N_1}; \dots ; \omega^{i_n, n}, \dots, \omega^{i_n, N_n}\} . \quad (4.9)$$

These one-forms determine a Lie algebra $\mathcal{G}(N_0, N_1, \dots, N_n)$, of dimension

$$\dim \mathcal{G}(N_0, \dots, N_n) = \sum_{p=0}^n (N_p - p + 1) \dim V_p , \quad (4.10)$$

under the conditions of the following

Theorem 3 Let $\mathcal{G} = V_0 \oplus V_1 \oplus \dots \oplus V_n$ be a splitting of \mathcal{G} into $n+1$ subspaces. Let \mathcal{G} fulfill the Weimar-Woods contraction condition (4.3) subordinated to this splitting, $c_{i_p, j_q}^{k_s} = 0$ if $s > p + q$. The one-form coefficients ω^{i_p, α_p} of (4.9) resulting from the expansion of the Maurer-Cartan forms ω^{i_p} in which $g^{i_p} \rightarrow \lambda^p g^{i_p}$, $p = 0, \dots, n$ (eq. (4.2)), determine Lie algebras $\mathcal{G}(N_0, N_1, \dots, N_n)$ of dimension (4.10) and structure constants

$$C_{i_p, \beta_p, j_q, \gamma_q}^{k_s, \alpha_s} = \begin{cases} 0, & \text{if } \beta_p + \gamma_q \neq \alpha_s \\ c_{i_p, j_q}^{k_s}, & \text{if } \beta_p + \gamma_q = \alpha_s \end{cases} \quad \begin{matrix} p, q, s = 0, 1, \dots, n \\ i_{p,q,s} = 1, 2, \dots, \dim V_{p,q,s} \\ \alpha_p, \beta_p, \gamma_p = p, p+1, \dots, N_p \end{matrix} ,$$

(eq.(4.8)) if $N_q = N_{q+1}$ or $N_q = N_{q+1} - 1$ ($q = 0, 1, \dots, n-1$) in (N_0, N_1, \dots, N_n) . In particular, the $N_p = p$ solution determines the algebra $\mathcal{G}(0, 1, \dots, n)$, which is the generalized İnönü-Wigner contraction of \mathcal{G} .

Proof.

To enforce the closure under d of the exterior algebra generated by the one-forms in (4.9) and to find the conditions that the various N_p must meet, we require, as in Sec. 3, that all the forms ω^{i_p, β_p} present in $d\omega^{k_s, \alpha_s}$ are already in (4.9). Looking at eqs. (A.7)-(A.10) and at the table above, we find the restrictions

| $\alpha_s \geq s$ | ω^{i_0, β_0} | ω^{i_1, β_1} | ω^{i_2, β_2} | \dots | ω^{i_n, β_n} |
|---------------------------|-------------------------|-------------------------|-------------------------|----------|-------------------------|
| $d\omega^{k_0, \alpha_0}$ | $N_0 \geq N_0$ | $N_1 \geq N_0$ | $N_2 \geq N_0$ | \dots | $N_n \geq N_0$ |
| $d\omega^{k_1, \alpha_1}$ | $N_0 \geq N_1 - 1$ | $N_1 \geq N_1$ | $N_2 \geq N_1$ | \dots | $N_n \geq N_1$ |
| $d\omega^{k_2, \alpha_2}$ | $N_0 \geq N_2 - 2$ | $N_1 \geq N_2 - 1$ | $N_2 \geq N_2$ | \dots | $N_n \geq N_2$ |
| \vdots | \vdots | \vdots | \vdots | \vdots | \vdots |
| $d\omega^{k_n, \alpha_n}$ | $N_0 \geq N_n - n$ | $N_1 \geq N_n - n + 1$ | $N_2 \geq N_n - n + 2$ | \dots | $N_n \geq N_n$ |

Closure conditions on the number N_p of one-forms ω^{i_p, α_p}

It then follows that there are 2^n types of solutions⁵ characterized by (N_0, N_1, \dots, N_n) , $N_p \geq p$, $p = 0, 1, \dots, n$, where

$$N_{q+1} = N_q \quad \text{or} \quad N_{q+1} = N_q + 1 \quad (q = 0, 1, \dots, n-1). \quad (4.11)$$

The JI for $\mathcal{G}(N_0, \dots, N_n)$,

$$C_{i_p, \beta_p}^{k_s, \alpha_s} C_{j_q, \gamma_q}^{i_p, \beta_p} C_{l_t, \rho_t m_u, \sigma_u} = 0 =$$

$$C_{i_p, \beta_p}^{k_s, \alpha_s} C_{j_q, \gamma_q}^{i_p, \beta_p} C_{l_t, \rho_t m_u, \sigma_u} + C_{i_p, \beta_p}^{k_s, \alpha_s} C_{j_q, \gamma_q}^{i_p, \beta_p} C_{l_t, \rho_t} + C_{i_p, \beta_p}^{k_s, \alpha_s} C_{l_t, \rho_t} C_{m_u, \sigma_u}^{i_p, \beta_p} C_{j_q, \gamma_q}, \quad (4.12)$$

are again satisfied through the JI for \mathcal{G} . This is a consequence of the fact that, for \mathcal{G} , the exterior derivative of the λ -expansion of the MC eqs. is the λ -expansion of their exterior derivative, but it may also be seen directly⁶.

A particular solution to (4.11) is obtained by setting $N_p = p$, $p = 0, 1, \dots, n$, which defines $\mathcal{G}(0, 1, \dots, n)$, with $\dim \mathcal{G}(0, 1, \dots, n) = \dim \mathcal{G} = r$ (from (4.10)). Since in this case α_p takes only one value ($\alpha_p = N_p = p$) for each $p = 0, 1, \dots, n$, we may drop this label. Then, the structure constants (4.8) for $\mathcal{G}(0, 1, \dots, n)$ read

$$C_{i_p j_q}^{k_s} = \begin{cases} 0, & \text{if } p + q \neq s \\ c_{i_p j_q}^{k_s}, & \text{if } p + q = s \end{cases} \quad \begin{matrix} p = 0, 1, \dots, n \\ i_{p,q,s} = 1, 2, \dots, \dim V_{p,q,s} \end{matrix}, \quad (4.13)$$

⁵ This number may be found, *e.g.* for $n = 3$, by writing symbolically the solution types in (4.11) as $[0,0,0,0]$ for $N_0 = N_1 = N_2 = N_3$; $[0,0,0,1]$ for $N_0 = N_1 = N_2, N_3 = N_2 + 1$; $[0,0,1,0]$ for $N_0 = N_1, N_2 = N_1 + 1 = N_3$; $[0,0,1,1]$ for $N_0 = N_1, N_2 = N_1 + 1, N_3 = N_2 + 1$; $[0,1,0,0]$ for $N_0, N_1 = N_0 + 1 = N_2 = N_3$; $[0,1,0,1]$ for $N_0, N_1 = N_0 + 1 = N_2, N_3 = N_2 + 1$; $[0,1,1,0]$ for $N_0, N_1 = N_0 + 1, N_2 = N_1 + 1 = N_3$ and $[0,1,1,1]$ for $N_0, N_1 = N_0 + 1, N_2 = N_1 + 1, N_3 = N_2 + 1$. This notation numbers the solutions in base 2; since $[0,1,1,1]$ corresponds to $2^3 - 1$ we see, adding $[0,0,0,0]$, that there are 2^3 ways of cutting the expansions that determine Lie algebras $\mathcal{G}(N_0, N_1, N_2, N_3)$, and 2^n in the general $\mathcal{G}(N_0, N_1, \dots, N_n)$ case.

⁶ We only need to check that (4.12) reduces to the JI for \mathcal{G} when the order in the upper index is the sum of those in the lower ones since the C 's are zero otherwise. First we see that, when $\alpha_s = \gamma_q + \rho_t + \sigma_u$, all three terms in the *r.h.s.* of (4.12) give non-zero contributions. This is so because the range of β_p is only limited by $\beta_p \leq \alpha_s$, which holds when $\beta_p = \rho_t + \sigma_u$, $\beta_p = \gamma_q + \rho_t$ and $\beta_p = \sigma_u + \gamma_q$. Secondly, and since $\beta_p \geq p$, we also need that the terms in the i_p sum that are suppressed in (4.12) when $p > \beta_p$ be also absent in the JI for \mathcal{G} so that (4.12) does reduce to the JI for \mathcal{G} . Consider *e.g.*, the first term in the *r.h.s.* of (4.12). If $p > \beta_p$, then $p > \rho_t + \sigma_u$ and hence $p > t + u$. Thus, by the W-W condition (4.3), this term will not contribute to the JI for \mathcal{G} and no sum over the subspace V_p index i_p will be lost as a result. The argument also applies to the other two terms for their corresponding β_p 's.

which shows that $\mathcal{G}(0, 1, \dots, n)$ is the generalized IW contraction of \mathcal{G} , in the sense of [11], subordinated to the splitting (4.1). Of course, when $n = 1$ ($p = 0, 1$), $V = V_0 \oplus V_1$, \mathcal{L}_0 is a subalgebra and eqs. (4.11) ((4.13)) reduce to (3.17) or (3.18) ((3.19)), *q.e.d.*

Since the structure of $\mathcal{G}(N_0, N_1, \dots, N_n)$ is fully predetermined by \mathcal{G} , we shall call $\mathcal{G}(N_0, N_1, \dots, N_n)$ an *expansion* of \mathcal{G} . For instance, for the case $\mathcal{G} = V_0 \oplus V_1 \oplus V_2$ there are four types of expanded algebras $\mathcal{G}(N_0, N_1, N_2)$ ⁷

$$N_0 = N_1 = N_2 \quad (4.14)$$

$$N_0 = N_1 = N_2 - 1, \quad (4.15)$$

$$N_0 = N_1 - 1 = N_2 - 1, \quad (4.16)$$

$$N_0 = N_1 - 1 = N_2 - 2. \quad (4.17)$$

Since in the above theorem $\alpha_p \geq p$ for all $p = 0, \dots, n$ was assumed, all types of one-forms ω^{i_p, α_p} with indexes i_p in all subspaces V_p were present in the basis of $\mathcal{G}(N_0, N_1, \dots, N_n)$. However, one may consider keeping terms in the expansion up to a certain order l , $l < n$ in which case due to (4.6), the forms ω^{i_p, α_p} with $p > l$ will not appear. Those with $p \leq l$ will determine the vector space $V^*(N_0, N_1, \dots, N_l)$ where N_l is the highest order l and hence α_l takes only the value $N_l = l = \alpha_l$. This vector space, of dimension

$$\dim V^*(N_0, \dots, N_l) = \sum_{p=0}^l (N_p - p + 1) \dim V_p, \quad (4.18)$$

determines a Lie algebra $\mathcal{G}(N_0, N_1, \dots, N_l)$ under the conditions of the following theorem

Theorem 4 Let $\mathcal{G} = \oplus_0^n V_p$, etc. as in Th. 3. Then, up to a certain order $N_l = l < n$, the one-forms

$$\{\omega^{i_0, \alpha_0}; \omega^{i_1, \alpha_1}; \dots; \omega^{i_l, \alpha_l}\} = \{\omega^{i_0, 0}, \omega^{i_0, 1}, \omega^{i_0, N_0+1}, \omega^{i_0, N_0}; \omega^{i_1, 1}, \omega^{i_1, N_1}, \omega^{i_1, N_1+1}; \dots; \omega^{i_l, N_l}\}, \quad (4.19)$$

where $N_l = l = \alpha_l$, determine a Lie algebra $\mathcal{G}(N_0, N_1, \dots, N_l)$ of dimension (4.18) and structure constants given by

$$C_{i_p, \beta_p, j_q, \gamma_q}^{k_s, \alpha_s} = \begin{cases} 0, & \text{if } \beta_p + \gamma_q \neq \alpha_s \\ c_{i_p, j_q}^{k_s}, & \text{if } \beta_p + \gamma_q = \alpha_s \end{cases} \quad \begin{matrix} p, q, s = 0, 1, \dots, l \\ i_{p,q,s} = 1, 2, \dots, \dim V_{p,q,s} \\ \alpha_p, \beta_p, \gamma_p = p, p+1, \dots, N_p; N_p \leq l \end{matrix}, \quad (4.20)$$

if $N_q = N_{q+1}$ or $N_q = N_{q+1} - 1$, ($q = 0, 1, \dots, l-1$).

Proof.

The restriction $\alpha_p \leq N_l = l < n$ on the order α_p of the one-forms ω^{i_p, α_p} implies, due to (4.6), that V_l is monodimensional and that $\omega^{i_l, l}$ is the last form entering (4.19). Then, looking at the closure conditions table in Th. 3, we can restrict ourselves to the box delimited by ω^{i_p, β_p} , $d\omega^{k_s, \alpha_s}$ with $p, s \leq l$. This box will give spaces $V^*(N_0, N_1, \dots, N_l)$, where $N_q = N_{q+1}$ or $N_q = N_{q+1} - 1$ ($q = 0, 1, \dots, l-1$), and these spaces will determine Lie algebras if the JI for (4.20)

$$C_{i_p, \beta_p, j_q, \gamma_q}^{k_s, \alpha_s} C_{l_t, \rho_t, m_u, \sigma_u}^{i_p, \beta_p} = 0, \quad i_{p,q,s} = 1, 2, \dots, \dim V_{p,q,s} \quad (4.21)$$

⁷With the notation of footnote 5, these correspond, respectively, to $[0,0,0]$, $[0,0,1]$, $[0,1,0]$ and $[0,1,1]$.

i. e., if $c_{i_p[j_q]c_{l_t m_u}}^{k_s} = 0$, $s, q, t, u \leq l$, is satisfied when $\alpha_s = \gamma_q + \rho_t + \sigma_u$ above. Note that this is not the JI for \mathcal{G} since i_p now runs over the basis of $\oplus_0^l V_p \subset \mathcal{G}$ only since $p \leq l$, and we are thus removing the values corresponding to the basis of $\oplus_{l+1}^n V_p$. However, if $p > l$ it is also *e.g.* $p > \beta_p = \rho_t + \sigma_u \geq t + u$ in which case $c_{l_t m_u}^{i_p} = 0$ by (4.3), *q.e.d.*

Since the structure constants (4.20) are obtained from those of \mathcal{G} by restricting the i_p indexes to be in the subspaces V_p , $p \leq l$, $\mathcal{G}(N_0, N_1, \dots, N_l)$ is *not* a subalgebra of $\mathcal{G}(N_0, N_1, \dots, N_n)$.

5 The expansion method for superalgebras

The above general procedure of generating Lie algebras from a given one does not lie on the antisymmetry of the structure constants of the original Lie algebra. Hence, with the appropriate changes to account for the grading, the method is applicable when \mathcal{G} is a Lie superalgebra, a case which we consider in this section.

Let G be a supergroup and \mathcal{G} its superalgebra. It is natural to consider a splitting of \mathcal{G} into the sum of three subspaces $\mathcal{G} = V_0 \oplus V_1 \oplus V_2$, V_1 being the fermionic part of \mathcal{G} and $V_0 \oplus V_2$ the bosonic part, so that the notation reflects the \mathbb{Z}_2 -grading of \mathcal{G} . The even space is always a subalgebra of \mathcal{G} but it may be convenient to consider it further split into the sum $V_0 \oplus V_2$ to allow for the case in which a subspace (V_0) of the bosonic space is itself a subalgebra \mathcal{L}_0 .

Notice that, since V_0 is a Lie algebra \mathcal{L}_0 , the \mathbb{Z}_2 -graduation of \mathcal{G} implies that the splitting $\mathcal{G} = V_0 \oplus V_1 \oplus V_2$ satisfies the W-W contraction conditions (4.3). Indeed, let $c_{i_p j_q}^{k_s}$ ($i_{p,q,s} = 1, \dots, \dim V_{p,q,s}$, $p, q, s = 0, 1, 2$) be the structure constants of \mathcal{G} . The \mathbb{Z}_2 -graduation of \mathcal{G} obviously implies

$$c_{i_0 j_0}^{k_1} = c_{i_0 j_1}^{k_2} = 0 \quad (5.1)$$

$$c_{i_0 j_1}^{k_0} = c_{i_1 j_1}^{k_1} = c_{i_0 j_2}^{k_1} = c_{i_2 j_1}^{k_0} = c_{i_2 j_1}^{k_2} = c_{i_2 j_2}^{k_1} = 0. \quad (5.2)$$

The first set of restrictions (5.1), together with the assumed subalgebra condition for V_0 (which, in addition, requires $c_{i_0 j_0}^{k_2} = 0$), are indeed the W-W conditions (4.3) for \mathcal{G} ; note that these conditions alone allow for $c_{i_1 j_1}^{k_0} \neq 0$, and $c_{i_1 j_1}^{k_2} \neq 0$ (and for $c_{i_1 j_1}^{k_1} \neq 0$, although here $c_{i_1 j_1}^{k_1} = 0$ due to the \mathbb{Z}_2 -grading).

To apply now the above general procedure one must rescale the group parameters. The rescaling (4.2) for $V = V_0 \oplus V_1 \oplus V_2$ takes the form

$$g^{i_0} \rightarrow g^{i_0}, \quad g^{i_1} \rightarrow \lambda g^{i_1}, \quad g^{i_2} \rightarrow \lambda^2 g^{i_2}. \quad (5.3)$$

Note that, if it proves convenient on dimensional grounds (a dimensionful parameter may be used to introduce dimensions in an originally dimensionless algebra), the redefinitions (5.3) may be changed. They are equivalent *e.g.*, to

$$g^{i_0} \rightarrow g^{i_0}, \quad g^{i_1} \rightarrow \mu^{1/2} g^{i_1}, \quad g^{i_2} \rightarrow \mu g^{i_2}, \quad (5.4)$$

with $\mu = \lambda^2$ obviously suggested by the \mathbb{Z}_2 -graded commutators; other redefinitions are equally possible.

The present \mathbb{Z}_2 -graded case fits into the preceding general discussion for $n = 2$, but with additional restrictions besides the W-W ones that follow from the \mathbb{Z}_2 -grading.

Theorem 5 Let $\mathcal{G} = V_0 \oplus V_1 \oplus V_2$ be a Lie superalgebra, V_1 its odd part, and $V_0 \oplus V_2$ the even one. Let further V_0 be a subalgebra \mathcal{L}_0 . As a result, \mathcal{G} satisfies the W-W conditions (4.3) and, further, V_1 is a symmetric coset. Then, the coefficients of the expansion of the Maurer-Cartan forms of \mathcal{G} rescaled by (5.3) determine Lie superalgebras $\mathcal{G}(N_0, N_1, N_2)$, $N_p \geq p$, $p = 0, 1, 2$, of dimension

$$\dim \mathcal{G}(N_0, N_1, N_2) = \left\lfloor \frac{N_0 + 2}{2} \right\rfloor \dim V_0 + \left\lfloor \frac{N_1 + 1}{2} \right\rfloor \dim V_1 + \left\lfloor \frac{N_2}{2} \right\rfloor \dim V_2 \quad , \quad (5.5)$$

and structure constants

$$C_{i_p, \beta_p, j_q, \gamma_q}^{k_s, \alpha_s} = \begin{cases} 0, & \text{if } \beta_p + \gamma_q \neq \alpha_s \\ c_{i_p j_q}^{k_s}, & \text{if } \beta_p + \gamma_q = \alpha_s \end{cases} \quad \begin{matrix} p, q, s = 0, 1, 2 \\ i_{p,q,s} = 1, 2, \dots, \dim V_{p,q,s} \\ \alpha_p, \beta_p, \gamma_p = p, p+2, \dots, N_p-2, N_p \end{matrix} \quad (5.6)$$

where⁸ $\lfloor \cdot \rfloor$ denotes integer part and the N_0, N_2 (even) and N_1 (odd) integers satisfy one of the three conditions below

$$N_0 = N_1 + 1 = N_2 \quad (5.7)$$

$$N_0 = N_1 - 1 = N_2 \quad , \quad (5.8)$$

$$N_0 = N_1 - 1 = N_2 - 2 \quad . \quad (5.9)$$

In particular, the superalgebra $\mathcal{G}(0, 1, 2)$ (eq. (5.9) for $N_0 = 0$) is the generalized İnönü-Wigner contraction of \mathcal{G} .

Proof. Since V_1 is a symmetric coset the rescaling (5.3) leads to an even (odd) power series in λ for the one-forms $\omega^{i_0}(g, \lambda)$ and $\omega^{i_2}(g, \lambda)$ ($\omega^{i_1}(g, \lambda)$), as in Sec. 3.1 (eqs. (3.23)). Thus, the conditions N_0, N_2 even, N_1 odd, have to be added to those that follow from the closure inequalities table in Th. 3. This gives the conditions

$$N_0 + 1 \geq N_1 \geq N_0 - 1 \quad (5.10)$$

$$N_1 + 1 \geq N_2 \geq N_1 - 1 \quad (5.11)$$

$$N_0 + 2 \geq N_2 \geq N_0 \quad , \quad (5.12)$$

from which eqs. (5.7)-(5.9) follow, *q.e.d.*

6 Application: the M-theory superalgebra

Let us apply these ideas to the case of the M-theory superalgebra (see [35][36, 37, 38, 39]). String/M theory implies that the $D=11$ supersymmetry algebra of nature may be one that includes the super-Poincaré algebra plus some additional ‘central’ bosonic generators. This (528+32+55)-dimensional algebra may be described by its MC equations

$$\begin{aligned} d\Pi^{\alpha\beta} &= -\frac{1}{4}(\gamma^{\mu\nu})^\alpha_\gamma \sigma_{\mu\nu} \wedge \Pi^{\gamma\beta} - \frac{1}{4}(\gamma^{\mu\nu})^\beta_\gamma \sigma_{\mu\nu} \wedge \Pi^{\gamma\alpha} - \pi^\alpha \wedge \pi^\beta \\ d\pi^\alpha &= -\frac{1}{4}(\gamma^{\mu\nu})^\alpha_\beta \sigma_{\mu\nu} \wedge \pi^\beta \\ d\sigma_{\mu\nu} &= -\sigma_\mu{}^\rho \wedge \sigma_{\rho\nu} \quad (\mu, \nu = 0, \dots, 10; \alpha, \beta = 1, \dots, 32) \quad , \end{aligned} \quad (6.1)$$

⁸For the rescaling (5.4) the orders would be $\alpha_p = \frac{p}{2}, \frac{p+2}{2}, \dots, \frac{N_p}{2}$.

where $\Pi^{\alpha\beta} = \Pi^{\beta\alpha}$ is a set of bosonic MC forms that may be expanded as⁹ $\Pi^{\alpha\beta} = -\frac{1}{32}(\Pi_\mu\gamma^\mu - \frac{1}{2}\Pi_{\mu\nu}\gamma^{\mu\nu} + \frac{1}{5!}\Pi_{\mu_1\dots\mu_5}\gamma^{\mu_1\dots\mu_5})^{\alpha\beta}$, π^α is a spinorial fermionic¹⁰ MC form and $\sigma_{\mu\nu}$ are the MC forms of the Lorentz generators. The spinor indexes α, β are raised and lowered with the $D = 11$ 32×32 skewsymmetric matrix $C_{\alpha\beta}$; the algebra (6.1) has dimension $560 + 55$. The M theory superalgebra is sometimes regarded (see *e.g.* [35]) as an IW contraction of the superalgebra $osp(1|32)$, of dimension 560, given by the 528+32 MC equations

$$\begin{aligned} d\rho^{\alpha\beta} &= -\rho^\alpha_\gamma \wedge \rho^{\gamma\beta} - \nu^\alpha \wedge \nu^\beta \\ d\nu^\alpha &= -\rho^\alpha_\beta \wedge \nu^\beta \quad (\alpha, \beta = 1, \dots, 32), \end{aligned} \quad (6.2)$$

with $\rho^{\alpha\beta} = \rho^{\beta\alpha}$ bosonic and ν^α fermionic or, using $\rho^{\alpha\beta} = -\frac{1}{32}(\rho_\mu\gamma^\mu - \frac{1}{2}\rho_{\mu\nu}\gamma^{\mu\nu} + \frac{1}{5!}\rho_{\mu_1\dots\mu_5}\gamma^{\mu_1\dots\mu_5})^{\alpha\beta}$ and taking γ matrices such that $\gamma^{\mu_1\dots\mu_{11}} = \epsilon^{\mu_1\dots\mu_{11}}$,

$$\begin{aligned} d\rho_\mu &= -\frac{1}{16}\rho_\nu \wedge \rho^\nu_\mu + \frac{1}{32(5!)^2}\epsilon^{\mu_1\dots\mu_{10}}{}_\mu \rho_{\mu_1\dots\mu_5} \wedge \rho_{\mu_6\dots\mu_{10}} - \nu^\alpha(\gamma_\mu)_{\alpha\beta} \wedge \nu^\beta \\ d\rho_{\mu\nu} &= -\frac{1}{16}\rho_\mu \wedge \rho_\nu - \frac{1}{16}\rho_{\mu\sigma} \wedge \rho^\sigma_\nu - \frac{1}{16(4!)}\rho_{\mu\mu_1\dots\mu_4} \wedge \rho^{\mu_4\dots\mu_1}_\nu - \nu^\alpha(\gamma_{\mu\nu})_{\alpha\beta} \wedge \nu^\beta \\ d\rho_{\mu_1\dots\mu_5} &= \frac{1}{16(5!)}\epsilon^{\sigma\nu_1\dots\nu_5}{}_{\mu_1\dots\mu_5} \rho_\sigma \wedge \rho_{\nu_1\dots\nu_5} + \frac{5}{16}\rho^\nu_{[\mu_1\dots\mu_4} \wedge \rho_{\mu_5]\nu} \\ &\quad + \frac{1}{4(4!)^2}\epsilon^{\nu_1\dots\nu_6}{}_{\mu_1\dots\mu_5} \rho_{\nu_1\nu_2\nu_3\sigma_1\sigma_2} \wedge \rho^{\sigma_2\sigma_1}{}_{\nu_4\nu_5\nu_6} - \nu^\alpha(\gamma_{\mu_1\dots\mu_5})_{\alpha\beta} \wedge \nu^\beta \\ d\nu^\alpha &= \frac{1}{32}\left(\rho_\mu\gamma^\mu - \frac{1}{2}\rho_{\mu\nu}\gamma^{\mu\nu} + \frac{1}{5!}\rho_{\mu_1\dots\mu_5}\gamma^{\mu_1\dots\mu_5}\right)^\alpha_\beta \wedge \nu^\beta. \end{aligned} \quad (6.3)$$

But this is so provided one *excludes* from (6.1) the 55 Lorentz generators $\sigma_{\mu\nu}$; otherwise, there are not enough generators in $osp(1|32)$ to give the M-algebra (6.1) by contraction.

We show now, however, that the expansion method allows us to obtain the M-theory superalgebra from $osp(1|32)$. Let us divide the $osp(1|32)$ vector space into three subspaces V_0 , V_1 and V_2 as in Sec. 5. Let then V_0^* be the space generated by the 55 MC forms $\rho^{\mu\nu} = \rho^{\alpha\beta}(\gamma^{\mu\nu})_{\alpha\beta}$ of the Lorentz subalgebra of $osp(1|32)$, V_1^* the fermionic subspace generated by ν^α , and V_2^* the space generated by the remaining 11+462 bosonic generators $\rho^\mu = \rho^{\alpha\beta}(\gamma^\mu)_{\alpha\beta}$, $\rho^{\mu_1\dots\mu_5} = \rho^{\alpha\beta}(\gamma^{\mu_1\dots\mu_5})_{\alpha\beta}$. Since, on the other hand, V_1 is a symmetric coset (Sec. 3.1), it follows that the expansions of the forms in V_0^* contain even powers of λ starting from λ^0 , that those of the forms in V_1^* include only odd powers in λ starting from λ^1 , and that those of V_2^* contain even orders starting with λ^2 , *i.e.*

$$V_0^* : \quad \rho^{\mu\nu} = \sum_{n=0}^{\infty} \lambda^{2n} \rho^{\mu\nu, 2n} \quad ; \quad (6.4)$$

$$V_1^* : \quad \nu^\alpha = \sum_{n=0}^{\infty} \lambda^{2n+1} \nu^{\alpha, 2n+1} \quad ; \quad (6.5)$$

$$V_2^* : \quad \rho^\mu = \sum_{n=1}^{\infty} \lambda^{2n} \rho^{\mu, 2n} \quad , \quad \rho^{\mu_1\dots\mu_5} = \sum_{n=1}^{\infty} \lambda^{2n} \rho^{\mu_1\dots\mu_5, 2n} \quad . \quad (6.6)$$

⁹Then, we have $\Pi_\mu = \Pi^{\alpha\beta}(\gamma_\mu)_{\alpha\beta}$, $\Pi_{\mu\nu} = \Pi^{\alpha\beta}(\gamma_{\mu\nu})_{\alpha\beta}$ and $\Pi_{\mu_1\dots\mu_5} = \Pi^{\alpha\beta}(\gamma_{\mu_1\dots\mu_5})_{\alpha\beta}$; this breaks the $GL(32, \mathbb{R})$ invariance of $\Pi^{\alpha\beta}$ down to $SO(1, 10)$. For an analysis that uses the maximal automorphism group $GL(32, \mathbb{R})$ of the M-algebra without the Lorentz part, see [47].

¹⁰The complex conjugation of a product is defined here as the product of the complex conjugates, without reversing the order.

Setting now $\lambda = \mu^{1/2}$, one may rewrite the series (6.4)-(6.6) as

$$\rho^{\alpha\beta} = \frac{1}{64}(\gamma_{\mu\nu})^{\alpha\beta}\rho^{\mu\nu,0} + \sum_{n=1}^{\infty} \mu^n \rho^{\alpha\beta,n} \quad (6.7)$$

$$\nu^\alpha = \sum_{n=0}^{\infty} \mu^{n+\frac{1}{2}} \nu^{\alpha,n+\frac{1}{2}}, \quad (6.8)$$

where $\rho^{\alpha\beta,n}$ for $n \geq 1$ collects the contributions from both (6.4), (6.6). We see now that, by Th. 5, we may keep powers up to $n = 1$ in (6.7) and $n = 0$ in (6.8), since this corresponds to taking $N_0 = N_2 = 2$, $N_1 = 1$ and, by eq. (5.5), $\dim \mathcal{G}(2, 1, 2) = 110 + 32 + 473 = 560 + 55$. So renaming $\nu^{\alpha,1/2} = \pi^\alpha$, $\rho^{\mu\nu,0} = 16\sigma^{\mu\nu}$, $\rho^{\alpha\beta,1} = \Pi^{\alpha\beta}$ and using eqs. (4.7) and (4.8) we obtain the M-algebra (6.1).

One may also consider higher orders in μ . A result, omitting the Lorentz part, was given in [42]. There, the expansion (6.7)-(6.8) was considered up to the power $\mu^{7/2}$ (without the term $n = 0$ for $\rho^{\mu\nu}$ which gives the Lorentz subalgebra), in an attempt to derive the M-algebra of Sezgin [39]. This superalgebra (not to be confused with the much smaller M-theory superalgebra of eq. (6.1)) was constructed generalizing earlier results in [33, 48] in order to re-interpret the FDA's associated to the WZ forms of supersymmetric extended objects and supergravity as Lie algebras (see also [34],[36]). The superalgebra obtained in [42] is not that in [39], but a subalgebra of it. Sezgin's superalgebras, and their associated enlarged superspace groups, are obtained by the third procedure in the Introduction: they are superalgebra and supergroup extensions [34].

It may be shown that, in contrast with Sezgin's M-algebra, the superalgebra in [42] is not enough to write the three- and six-forms of the $D = 11$ membrane and five-brane in flat superspace in terms of invariant one-forms. In fact, the extended algebras considered in [48, 34] cannot be obtained by the expansion method but for some exceptions (for example, it may be seen that the Green algebra [49] in three spacetime dimensions may be obtained from $osp(1|2)$). Nevertheless, we have seen here that the Lorentz part of the M-theory superalgebra, which is missing in the IW contraction of $osp(1|32)$, may be generated when suitable orders of the series expansions of its MC forms are retained: the M-algebra (6.1) is $osp(1|32)(2, 1, 2)$.

To conclude, we mention that another M-type algebra having the 528-dimensional $sp(32)$ as its automorphism group, namely

$$\begin{aligned} d\Pi^{\alpha\beta} &= -\sigma^\alpha_\gamma \wedge \Pi^{\gamma\beta} - \sigma^\beta_\gamma \wedge \Pi^{\gamma\alpha} - \pi^\alpha \wedge \pi^\beta \\ d\pi^\alpha &= -\sigma^\alpha_\gamma \wedge \pi^\gamma \\ d\sigma^{\alpha\beta} &= -\sigma^\alpha_\gamma \wedge \sigma^{\gamma\beta}, \end{aligned} \quad (6.9)$$

may also be obtained using the natural splitting of $osp(1|32)$ in its even and odd parts. In the notation of Sec. 3.1 (expansion in λ) the superalgebra (6.9) is $osp(1|32)(2,1)$, where now $\rho^{\alpha\beta,0} = \sigma^{\alpha\beta}$ corresponds to the $sp(32)$ subalgebra, $\rho^{\alpha\beta,2} = \Pi^{\alpha\beta}$ and $\nu^{\alpha,1} = \pi^\alpha$.

7 Extension to gauge free differential (super)algebras and Chern-Simons theories

We have shown that, by rescaling some group variables and identifying equal powers of λ in the MC equations of an algebra \mathcal{G} , one obtains the MC equations (4.7) for the algebra

$\mathcal{G}(N_0, N_1, \dots, N_n)$. Using these MC equations we may now construct the corresponding gauge free differential (super)algebras (for FDA see [43, 36, 44, 27]).

Let us examine the general case of Sec. 4. To obtain a gauge FDA we replace the MC forms ω^{k_s, α_s} by the gauge field (or ‘soft’, see [27]) one-forms A^{k_s, α_s} and introduce their corresponding curvatures F^{k_s, α_s} by (cf. (4.7))

$$F^{k_s, \alpha_s} = dA^{k_s, \alpha_s} + \frac{1}{2} C_{i_p, \beta_p j_q, \gamma_q}^{k_s, \alpha_s} A^{i_p, \beta_p} \wedge A^{j_q, \gamma_q} := DA^{k_s, \alpha_s} \quad , \quad (7.1)$$

where the $C_{i_p, \beta_p j_q, \gamma_q}^{k_s, \alpha_s}$ are defined as in (4.8). The curvatures F^{k_s, α_s} satisfy the consistency conditions expressed by the Bianchi identities

$$dF^{k_s, \alpha_s} = C_{i_p, \beta_p j_q, \gamma_q}^{k_s, \alpha_s} F^{i_p, \beta_p} \wedge A^{j_q, \gamma_q} \quad , \quad (DF = 0) \quad . \quad (7.2)$$

The Cartan structure equations (7.1) and the Bianchi identities (7.2) define the gauge FDA associated with $\mathcal{G}(N_0, N_1, \dots, N_n)$.

One may look at eqs. (7.1)-(7.2) as coming from expansions of the type (4.4)-(4.6) of the original gauge FDA for \mathcal{G} ,

$$F^{k_s} = dA^{k_s} + \frac{1}{2} c_{i_p j_q}^{k_s} A^{i_p} \wedge A^{j_q} \quad , \quad dF^{k_s} = c_{i_p j_q}^{k_s} F^{i_p} \wedge A^{j_q} \quad , \quad (7.3)$$

or directly as defining the gauge FDA associated with the $\mathcal{G}(N_0, N_1, \dots, N_n)$ Lie algebra. Moreover, the infinitesimal gauge transformations of the A^{k_s, α_s} , F^{k_s, α_s} , with parameters φ^{k_s, α_s} corresponding to the expanded group $G(N_0, N_1, \dots, N_n)$, are given by

$$\delta A^{k_s, \alpha_s} = d\varphi^{k_s, \alpha_s} - C_{i_p, \beta_p j_q, \gamma_q}^{k_s, \alpha_s} \varphi^{i_p, \beta_p} A^{j_q, \gamma_q} \quad , \quad \delta F^{k_s, \alpha_s} = C_{i_p, \beta_p j_q, \gamma_q}^{k_s, \alpha_s} F^{i_p, \beta_p} \varphi^{j_q, \gamma_q} \quad , \quad (7.4)$$

recalling that the original gauge fields A^{k_s} of G transform as

$$\delta A^{k_s} = d\varphi^{k_s} - c_{i_p j_q}^{k_s} \varphi^{i_p} A^{j_q} \quad . \quad (7.5)$$

We note that the above FDA algebras are *contractible* [43, 44, 27] since they are generated by pairs of forms (A, B) such that $B = dA$ and $dB = 0$, where A corresponds to A^{k_s, α_s} and $B = dA = F - A^2$ (i.e., $F^{k_s, \alpha_s} - \frac{1}{2} C_{i_p, \beta_p j_q, \gamma_q}^{k_s, \alpha_s} A^{i_p, \beta_p} \wedge A^{j_q, \gamma_q}$). Thus, the FDA’s de Rham cohomology is trivial and every closed form in the FDA may be written as d of a form constructed from its generators.

Let us now see how the above new gauge FDA’s may be used to obtain Chern-Simons (CS) gauge theories. Given a Lie algebra \mathcal{G} , a Chern-Simons (CS) field theory is, generically, one for which the Lagrangian form is the potential of a $2l$ -form $H = \langle F^{I_1}, \dots, F^{I_l} \rangle$ constructed from the curvature two-forms F^I in the following way. Let $k_{I_1 \dots I_l}$ be a (graded)symmetric invariant tensor on \mathcal{G} , where the index I runs over the values of the \mathcal{G} basis index. Then, the CS Lagrangian is a $(2l - 1)$ -form B such that

$$H = dB = \langle F^{I_1}, \dots, F^{I_l} \rangle = k_{I_1 \dots I_l} F^{I_1} \wedge \dots \wedge F^{I_l} \quad ; \quad (7.6)$$

as a result, the CS structure is intrinsically odd dimensional. Since the *r.h.s.* is gauge invariant, the CS form B is gauge quasi-invariant i.e. its gauge transformation is given by the differential of a $(2l - 2)$ -form¹¹. When \mathcal{G} is a classical algebra (or one of its \mathbb{Z}_2 -graded

¹¹For Chern-Weil invariants and for explicit expressions of CS forms and their gauge transformation properties see *e.g.*, [23].

counterparts), one may write $F = F^I T_I$, T_I being a matrix realization of the basis of \mathcal{G} . Then, when different from zero, the (graded) symmetrized trace $\text{sTr}(T_{I_1} \dots T_{I_l})$ gives a G -invariant symmetric tensor, so that

$$dB = \text{sTr}(F \wedge \dots \wedge F) . \quad (7.7)$$

Consider the case of a CS theory for the algebra $\mathcal{G}(N_0, N_1, \dots, N_n)$. Let A^I and F^I be the gauge and curvature forms of the gauge FDA associated with $\mathcal{G} = \oplus_0^n V_p$, so that now $I = i_1, \dots, i_n$ as in Th. 3. If $k_{I_1 \dots I_l}$ is a (graded)symmetric invariant tensor of rank l on \mathcal{G} , the CS action associated with \mathcal{G} is given by the integral over a $(2l-1)$ -dimensional manifold \mathcal{M}^{2l-1} of a potential form of $k_{I_1 \dots I_l} F^{I_1} \wedge \dots \wedge F^{I_l}$. Inserting in (7.6) the expansion of the gauge forms ($\omega^{k_s, \alpha_s} \rightarrow A^{k_s, \alpha_s}$) one finds¹²

$$I[A, \lambda] = \int_{\mathcal{M}^{2l-1}} B(A, \lambda) = \int_{\mathcal{M}^{2l-1}} \sum_{\alpha=0}^{\infty} \lambda^\alpha B_\alpha(A) = \sum_{\alpha=0}^{\infty} \lambda^\alpha I_\alpha[A] . \quad (7.8)$$

For each order N , one obtains a CS action that is invariant under the transformations (7.4), because it is the integral of a form the differential of which is the coefficient of λ^N in the expansion of the invariant form $H = \text{sTr}(F \wedge \dots \wedge F)$, and these coefficients are separately invariant. Once an order N is fixed, a gauge FDA algebra is selected naturally: it is the one containing all the gauge fields and curvatures that appear in $dB_N(A)$ in agreement with Th. 3 in Sec. 4, since this guarantees the consistency of (7.1)-(7.2). In the case that we are considering, the fields F^{k_s, α_s} appear in the coefficient $dB_N(A)$ so that $N_p \leq N$ for all p . The action $I_N[A]$ in (7.8) could have been obtained directly by using the corresponding symmetric $G(N_0, \dots, N_n)$ -invariant form.

Let us assume that \mathcal{G} is simple. Then, with dimensionless structure constants for \mathcal{G} , it is not possible to assign consistently physical dimensions to its generators. The fields A^I are also dimensionless and the corresponding CS integral cannot have dimensions of an action. One may, however, rescale the fields A^I as in a generalized W-W contraction (in our scheme corresponding to $\mathcal{G}(0, 1, \dots, n)$), $A^{i_p} \rightarrow \lambda^p A^{i_p, p}$, and declare that λ has some definite physical dimensions. The resulting action is constructed in terms of dimensionful fields, at the price of introducing an explicit dependence of the structure constants on a dimensionful parameter, which disappears by conveniently taking the limit $\lambda \rightarrow 0$. This process gives a CS theory on the contracted algebra when the gauge fields have suitable physical dimensions. But the expansion method is more general, and we may obtain true actions with the right physical dimensions, by using (7.8), due to the fact that the action $I_N[A]$ has dimensions $[\lambda]^{-N}$. Moreover, in contrast with a contraction, the expansion method gives a CS theory for a Lie (super)algebra that, in general, is of higher dimension than that of \mathcal{G} . We now illustrate both procedures, the contraction and the expansion one, using the case of three-dimensional supergravity as an example.

¹²We shall ignore here the coupling constant and its possible quantization (see [50]). For the case *e.g.*, of odd-dimensional CS gravities, see [51]; the quantization may result from a mechanism similar to that associated with Wess-Zumino-Witten-Novikov terms.

8 Application to Chern-Simons gauge theory of supergravity

It is well known that (super)gravities in three spacetime dimensions are CS gauge theories [52, 53, 54] and hence topological and exactly solvable [54], which allows for the construction of an exact quantum theory. For instance, Poincaré supergravity in three spacetime dimensions is a CS theory for the eight-dimensional $D=3$ superPoincaré Lie algebra defined by the MC equations

$$\begin{aligned} d\sigma^{ab} &= -\sigma^a{}_c \wedge \sigma^{cb} \\ d\pi^\alpha &= -\frac{1}{4}\sigma_{ab}(\gamma^{ab})^\alpha{}_\beta \wedge \pi^\beta \\ d\Pi^a &= -\sigma^a{}_b \wedge \Pi^b - \pi^\alpha \gamma^a_{\alpha\beta} \wedge \pi^\beta \quad (a, b = 0, 1, 2, \alpha, \beta = 1, 2), \end{aligned} \quad (8.1)$$

where Π^a , π^α and $\sigma_{ab} = -\sigma_{ba}$ are, respectively, the translations, supertranslations and Lorentz MC forms. The gauge FDA corresponding to (8.1), of gauge one-forms ω_{ab} , ψ^α and e^a (corresponding, respectively, to the MC forms σ_{ab} , π^α , Π^a) and curvatures R_{ab} , \mathcal{T}^α and T^a , is

$$\begin{aligned} R_{ab} &= d\omega_{ab} + \omega_{ac} \wedge \omega^c{}_b := D\omega_{ab} \quad , \\ DR_{ab} &= 0 \quad ; \\ \mathcal{T}^\alpha &= d\psi^\alpha + \frac{1}{4}\omega_{ab}(\gamma^{ab})^\alpha{}_\beta \wedge \psi^\beta := D\psi^\alpha \quad , \\ D\mathcal{T}^\alpha &= \frac{1}{4}R_{ab}(\gamma^{ab})^\alpha{}_\beta \wedge \psi^\beta \quad ; \\ T^a &= de^a + \omega^a{}_b \wedge e^b + \psi^\alpha \gamma^a_{\alpha\beta} \wedge \psi^\beta := De^a + \psi^\alpha \gamma^a_{\alpha\beta} \wedge \psi^\beta \quad ; \\ DT^a &= R^a{}_b \wedge e^b + 2\mathcal{T}^\alpha \gamma^a_{\alpha\beta} \wedge \psi^\beta \quad . \end{aligned} \quad (8.2)$$

The action is then given by

$$I = \int_{\mathcal{M}^3} (\epsilon^{abc} R_{ab} \wedge e_c + 4\psi_\alpha \wedge \mathcal{T}^\alpha), \quad (8.3)$$

where we take $\gamma^{abc} = \epsilon^{abc}$. The integrand is a potential form of the closed gauge-invariant 4-form $\epsilon^{abc} R_{ab} \wedge T_c + 4\mathcal{T}_\alpha \wedge \mathcal{T}^\alpha$. Eq. (8.3) is the action for Poincaré supergravity in $2+1$ dimensions [55] written using the first order gauge formulation (see *e.g.*, [27]), the field equations of which are $T^a = 0$ (equations for ω ; metricity condition), $R^{ab} = 0$ (equations for e ; flat space Einstein equations) and $\mathcal{T}^\alpha = D\psi^\alpha = 0$ (equations for ψ or three-dimensional counterpart of the Rarita-Schwinger equations).

The superalgebra (8.1) can be viewed as a contraction of the Lie superalgebra $osp(1|2) \oplus osp(0|2) = osp(1|2) \oplus sp(2)$, also known as the type (1,0) anti-de Sitter superalgebra in $D = 2+1$ [53]. Since the algebra is the direct sum of two simple ones, no physical dimensions can be assigned to the forms. However one may rescale some of them using a dimensionful scale, which then appears explicitly in the structure constants, and these rescaled generators have dimensions. From the corresponding CS integral $I(\lambda)$ one may construct the action $I(\lambda)/\lambda$, with dimensions of length (those of an action in three dimensions in geometrized units), which is the action for supergravity with a negative cosmological constant. The limit $\lambda \rightarrow 0$

for $I(\lambda)/\lambda$ turns out to be well defined, and gives (8.3). Explicitly, $osp(1|2) \oplus sp(2)$ may be given by the MC equations

$$\begin{aligned} d\sigma^{ab} &= -\sigma^a{}_c \wedge \sigma^{cb} - \tilde{\Pi}^a \wedge \tilde{\Pi}^b - \tilde{\pi}^\alpha (\gamma^{ab})_{\alpha\beta} \wedge \tilde{\pi}^\beta \\ d\tilde{\pi}^\alpha &= -\frac{1}{4}\sigma_{ab}(\gamma^{ab})^\alpha{}_\beta \wedge \tilde{\pi}^\beta + \frac{1}{2}\tilde{\Pi}_a(\gamma^a)^\alpha{}_\beta \wedge \tilde{\pi}^\beta \\ d\tilde{\Pi}^a &= -\sigma^a{}_b \wedge \tilde{\Pi}^b - \tilde{\pi}^\alpha \gamma_{\alpha\beta}^a \wedge \tilde{\pi}^\beta \quad (a, b, c = 0, 1, 2, \alpha, \beta = 1, 2) \quad , \end{aligned} \quad (8.4)$$

where, again, $\sigma^{ab} = -\sigma^{ba}$ and $\tilde{\Pi}^a$ are bosonic and $\tilde{\pi}^\alpha$ is fermionic. Starting from this algebra one may obtain the gauge FDA by using eq. (7.3) for the gauge forms $(\omega_{ab}, \tilde{\psi}^\alpha, \tilde{e}^a)$ and curvatures $(R_{ab}, \tilde{T}^\alpha, \tilde{T}^a)$. These fields cannot be assigned physical dimensions unless some of them are rescaled. The obvious choice is to set $\tilde{\Pi}^a = \mu\Pi^a$, $\tilde{\pi}^\alpha = \mu^{1/2}\pi^\alpha$, and hence $\tilde{e}^a = \mu e^a$, $\tilde{\psi}^\alpha = \mu^{1/2}\psi^\alpha$, $\tilde{T}^a = \mu T^a$, $\tilde{T}^\alpha = \mu^{1/2}\mathcal{T}^\alpha$ where the parameter μ has dimensions $[\mu] = L^{-1}$ so that the algebra then reads

$$\begin{aligned} d\sigma^{ab} &= -\sigma^a{}_c \wedge \sigma^{cb} - \mu^2\Pi^a \wedge \Pi^b - \mu\pi^\alpha (\gamma^{ab})_{\alpha\beta} \wedge \pi^\beta \\ d\pi^\alpha &= -\frac{1}{4}\sigma_{ab}(\gamma^{ab})^\alpha{}_\beta \wedge \pi^\beta + \frac{\mu}{2}\Pi_a(\gamma^a)^\alpha{}_\beta \wedge \pi^\beta \\ d\Pi^a &= -\sigma^a{}_b \wedge \Pi^b - \pi^\alpha \gamma_{\alpha\beta}^a \wedge \pi^\beta \quad , \end{aligned} \quad (8.5)$$

and the contraction limit $\mu \rightarrow 0$ reproduces (8.1). The associated gauge FDA ($\sigma^{ab} \rightarrow \omega^{ab}$, $\Pi_a \rightarrow e_a$, $\pi^\alpha \rightarrow \psi^\alpha$) is

$$\begin{aligned} R_{ab} &= d\omega_{ab} + \omega_{ac} \wedge \omega^c{}_b + \mu^2 e_a \wedge e_b + \mu\psi^\alpha (\gamma_{ab})_{\alpha\beta} \wedge \psi^\beta \quad , \\ DR_{ab} &= \mu^2 T_a \wedge e_b - \mu^2 e_a \wedge T_b + 2\mu\mathcal{T}^\alpha (\gamma_{ab})_{\alpha\beta} \wedge \psi^\beta \quad ; \\ \mathcal{T}^\alpha &= d\psi^\alpha + \frac{1}{4}\omega_{ab}(\gamma^{ab})^\alpha{}_\beta \wedge \psi^\beta - \frac{\mu}{2}e_a(\gamma^a)^\alpha{}_\beta \wedge \psi^\beta := D\psi^\alpha - \frac{\mu}{2}e_a(\gamma^a)^\alpha{}_\beta \wedge \psi^\beta \quad , \\ D\mathcal{T}^\alpha &= \frac{1}{4}R_{ab}(\gamma^{ab})^\alpha{}_\beta \wedge \psi^\beta - \frac{\mu}{2}T_a(\gamma^a)^\alpha{}_\beta \wedge \psi^\beta + \frac{\mu}{2}e_a(\gamma^a)^\alpha{}_\beta \wedge \mathcal{T}^\beta \quad ; \\ T^a &= de^a + \omega^a{}_b \wedge e^b + \psi^\alpha \gamma_{\alpha\beta}^a \wedge \psi^\beta := De^a + \psi^\alpha \gamma_{\alpha\beta}^a \wedge \psi^\beta \quad , \\ DT^a &= R^a{}_b \wedge e^b + 2\mathcal{T}^\alpha \gamma_{\alpha\beta}^a \wedge \psi^\beta \quad . \end{aligned} \quad (8.6)$$

One may construct a three-dimensional CS theory starting from the gauge invariant four-form $\epsilon^{abc}R_{ab} \wedge \tilde{T}_c + 4\tilde{T}_\alpha \wedge \tilde{T}^\alpha = \mu\epsilon^{abc}R_{ab} \wedge T_c + 4\mu\mathcal{T}_\alpha \wedge \mathcal{T}^\alpha$. Since the gauge algebra is contractible, the CS integral is then easily found to be

$$I(\mu) = \mu \int_{\mathcal{M}^3} \left(\epsilon^{abc}R_{ab} \wedge e_c + 4\psi_\alpha \wedge \mathcal{T}^\alpha - \frac{2}{3}\mu^2\epsilon^{abc}e_a \wedge e_b \wedge e_c + 2\mu\psi^\alpha (\gamma_a)_{\alpha\beta} \wedge \psi^\beta \wedge e^a \right) , \quad (8.7)$$

which gives the $(1, 0)$ AdS supergravity lagrangian in differential form, a supersymmetrization of $D = 3$ gravity with negative cosmological constant. Taking the $\mu \rightarrow 0$ limit in $I(\mu)/\mu$, the CS supergravity action (8.3) is recovered.

It is worth clarifying at this stage the nature of the above contraction. It is performed by writing $osp(1|2) \oplus sp(2)$ in a ('pseudoextended') form that disguises its actual direct (trivial) sum structure, which may be recovered by making the change of basis

$$\rho'^{\alpha\beta} = \frac{1}{4}\gamma_a^{\alpha\beta}(\epsilon^{abc}\sigma_{bc} + 2\tilde{\Pi}^a) \quad , \quad \rho^{\alpha\beta} = \frac{1}{4}\gamma_a^{\alpha\beta}(\epsilon^{abc}\sigma_{bc} - 2\tilde{\Pi}^a) \quad , \quad \nu^\alpha = \sqrt{2}\tilde{\pi}^\alpha \quad , \quad (8.8)$$

which exhibits the explicitly direct sum $osp(1|2) \oplus sp(2)$ form,

$$\begin{aligned} d\rho^{\alpha\beta} &= -\rho^\alpha_\gamma \wedge \rho^{\gamma\beta} - \nu^\alpha \wedge \nu^\beta \quad , \\ d\nu^\alpha &= -\rho^\alpha_\beta \wedge \nu^\beta \quad ; \\ d\rho'^{\alpha\beta} &= -\rho'^\alpha_\gamma \wedge \rho^{\gamma\beta} \quad (\alpha, \beta = 1, 2) . \end{aligned} \quad (8.9)$$

The contraction of (8.5) does not respect the above direct sum structure and, by not doing so, generates non-trivial cohomology; this is why the $D = 3$ superPoincaré algebra may be obtained. This example is not unique. For instance, some gauge formulations of $D=1+1$ gravity are based on a four-dimensional central extension of the $(1+1)$ -Poincaré algebra with a ‘magnetic’ modification of the momenta commutators. This algebra is obtained by means of a so called ‘unconventional’ contraction [56, 57] of (the de Sitter or anti de Sitter algebra) $so(2, 1)$. This corresponds, actually, to making a standard IW contraction of a trivial extension of $so(2, 1)$ by a one-dimensional algebra in such a way that the contracted algebra becomes a non-trivial extension of the Poincaré one. Again, this procedure corresponds to transforming [5, 24] a two-coboundary (that on $so(2, 1)$, giving its trivial extension) into a non trivial two-cocycle (on the $(1 + 1)$ -Poincaré) by the contraction limit (see *e.g.*, [23] for the cohomology that governs extension theory). Thus, these are all examples of the first method mentioned in the Introduction, the contraction one, which preserves the dimension of the algebra.

We now turn to the expansion method. Instead of using the *eight*-dimensional $osp(1|2) \oplus sp(2)$ algebra to obtain the $D=3$ superPoincaré by an IW contraction, we now take the *five*-dimensional $osp(1|2)$ MC one as the starting point. This superalgebra is also the de Sitter algebra in two dimensions. Its MC eqs. are given by the first two equations in (8.9) (cf. (6.2)). We immediately see that the $D = 3$ superPoincaré algebra is $osp(1|2)(2, 1)$, of dimension $= 2\dim V_0 + \dim V_1 = 8$ (eq. (5.5)), making the identifications $\rho^{\alpha\beta, 0} = \frac{1}{4}(\gamma^{ab})^{\alpha\beta}\sigma_{ab}$, $\nu^{\alpha, 1} = \pi^\alpha$ and $\rho^{\alpha\beta, 2} = -\frac{1}{2}\Pi_a(\gamma^a)^{\alpha\beta}$ (the orders refer here to powers of λ , not μ ; note also that in $D = 3$ either $\gamma_{\alpha\beta}^a$ or their duals $\gamma_{\alpha\beta}^{ab}$ provide a basis for the symmetric tensors). The $osp(1|2)$ gauge FDA is generated by the gauge forms $f^{\alpha\beta}$, ξ^α of curvatures $\Omega^{\alpha\beta}$, Ψ^α , and is given by

$$\begin{aligned} \Omega^{\alpha\beta} &= df^{\alpha\beta} + f^\alpha_\gamma \wedge f^{\gamma\beta} + \xi^\alpha \wedge \xi^\beta \\ d\Omega^{\alpha\beta} &= \Omega^\alpha_\gamma \wedge f^{\gamma\beta} - f^\alpha_\gamma \wedge \Omega^{\gamma\beta} + \Psi^\alpha \wedge \xi^\beta - \xi^\alpha \wedge \Psi^\beta \\ \Psi^\alpha &= d\xi^\alpha + f^\alpha_\beta \wedge \xi^\beta \\ d\Psi^\alpha &= \Omega^\alpha_\beta \wedge \xi^\beta - f^\alpha_\beta \wedge \Psi^\beta \quad (\alpha, \beta = 1, 2) \quad , \end{aligned} \quad (8.10)$$

where $f^{\alpha\beta}$, $\Omega^{\alpha\beta}$ are even and symmetric in α, β , and ξ^α , Ψ^α are fermionic. The indexes are raised and lowered by the 2×2 antisymmetric matrix $\epsilon_{\alpha\beta}$, $\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta$ and so on. The corresponding gauge transformations, for parameters $\Lambda^{\alpha\beta} = \Lambda^{\beta\alpha}$, φ^α corresponding to $f_{\alpha\beta}$ and ξ^α respectively are

$$\begin{aligned} \delta f^{\alpha\beta} &= d\Lambda^{\alpha\beta} - \Lambda^\alpha_\gamma f^{\gamma\beta} + f^\alpha_\gamma \Lambda^{\gamma\beta} - \xi^\alpha \varphi^\beta + \varphi^\alpha \xi^\beta \\ \delta \xi^\alpha &= d\varphi^\alpha - \Lambda^\alpha_\beta \xi^\beta + f^\alpha_\beta \varphi^\beta \quad . \end{aligned} \quad (8.11)$$

Let us now use the expansion method to obtain the CS supergravity action in $D = 2+1$ from a CS integral for $osp(1|2)$. The CS integral based on (8.10) is constructed from the gauge invariant closed four-form

$$H = \Omega^\alpha_\beta \wedge \Omega^\beta_\alpha - 2\Psi_\alpha \wedge \Psi^\alpha \quad . \quad (8.12)$$

We look for an integral form with dimensions of an action *i.e.*, of length in geometrized units. If μ is the expansion parameter, and $[\mu] = L^{-1}$, we need the order one in μ of H . The present situation is that of Sec. 5 with $V_2^* = 0$ and where V_0^* and V_1^* are generated by $\rho^{\alpha\beta}$ and ν^α respectively. Since V_1 is a symmetric coset, equation (3.23) for $\lambda = \mu^{1/2}$ applies and we have to consider the following expansions

$$f^{\alpha\beta} = \sum_{n=0}^{\infty} f^{\alpha\beta,n} \mu^n, \quad \xi^\alpha = \sum_{n=0}^{\infty} \xi^{\alpha,n+1/2} \mu^{n+1/2}, \quad (8.13)$$

and similarly for the curvatures. We may obtain different FDA gauge superalgebras by retaining different orders according to Th. 5 for $V_2 = 0$ (or to (3.29)-(3.30)). On the other hand, the fact that to construct the action we need the term proportional to μ in the expansion of H requires $n = 1, n = 0$ for the upper limits of the sums in (8.13) (these correspond to the $N_0 = 2, N_1 = 1$ that characterize $osp(1|2)(2,1)$), in agreement with (5.7) and (3.30). So the relevant algebra will correspond to the forms $f^{\alpha\beta,0}, f^{\alpha\beta,1}, \xi^{\alpha,1/2}$,

$$f^{\alpha\beta,0} = \frac{1}{4}(\gamma^{ab})^{\alpha\beta} \omega_{ab}, \quad f^{\alpha\beta,1} = -\frac{1}{2}(\gamma^a)^{\alpha\beta} e_a, \quad \xi^{\alpha,1/2} = \psi^\alpha. \quad (8.14)$$

Then the resulting gauge FDA is precisely (8.2), and the term proportional to μ in the expansion of H in (8.12) is the closed form $\frac{1}{2}\gamma^{abc}R_{ab} \wedge T_c - 2\mathcal{T}_\alpha \wedge \mathcal{T}^\alpha$, the potential form of which leads to (8.3). This translates the fact that the $D = 3$ superPoincaré algebra is $osp(1|2)(2,1)$.

The same procedure may be applied to obtain a CS theory based on $osp(1|32)$ (see [58, 59, 60, 61, 62]), using either the splitting of Th. 5 or one with $V_2 = 0$, in which case V_0 and V_1 are simply the bosonic and fermionic parts of the superalgebra (as used at the end of Sec. 6). The resulting algebras have a semidirect structure, where the Lorentz ($sp(32)$) algebra is the simple factor in the algebra resulting from the first (second) splitting. Results on the corresponding $D = 11$ CS theory will be published separately.

9 Conclusions and outlook

In this paper we have described the expansion method, a procedure of obtaining new (super)algebras $\mathcal{G}(N_0, \dots, N_p)$ from a given one \mathcal{G} that we denote *expansions* of \mathcal{G} (Th.3). It is based in the power expansion of the MC equations that results from rescaling certain group variables. These expansions are in principle infinite, but some truncations are consistent and define the Maurer-Cartan equations of new (super)algebras, the structure constants of which are obtained from those of the original algebra \mathcal{G} . We have considered the different possible $\mathcal{G}(N_0, \dots, N_p)$ algebras subordinated to various splittings of \mathcal{G} and discussed their structure. We have seen that in some cases (when the splitting of \mathcal{G} satisfies the Weimar-Woods conditions) the resulting algebras include the simple or generalized İnönü-Wigner contractions of \mathcal{G} , but that this is not always the case. They may be, however, IW contractions of certain higher-dimensional algebras related to the original one, as we have seen in Sec. 8. In general, the new ‘expanded’ algebras have higher dimension than the original one.

Since \mathcal{G} is the only ingredient of the expansion method, it is clear that the extension procedure (which involves *two* algebras) is richer when one is looking for new (super)algebras, as discussed at the end of Sec. 6. As it is the case for contractions, the expansion method

is more constrained. Nevertheless, we have used it to obtain the M-theory superalgebra, including its Lorentz part, from $osp(1|32)$. After formally extending the method to the case of gauge free differential algebras, we have applied it to the case of CS supergravity in $2+1$ dimensions where, using that $D=3$ superPoincaré is $osp(1|2)(2,1)$, we have recovered the Chern-Simons supergravity action from a CS form for $osp(1|2)$. The application of the expansion method to the $D = 11$ case will be presented elsewhere.

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A Appendix: expansion of $d\omega^{k_s, \alpha_s}$

Inserting (4.4)-(4.6) into (2.10) where now $p, q, s = 0, 1, \dots, n$, and using

$$\left(\sum_{\alpha=p}^{\infty} \lambda^{\alpha} \omega^{i_p, \alpha} \right) \wedge \left(\sum_{\alpha=q}^{\infty} \lambda^{\alpha} \omega^{j_q, \alpha} \right) = \sum_{\alpha=p+q}^{\infty} \lambda^{\alpha} \sum_{\beta=p}^{\alpha-q} \omega^{i_p, \beta} \wedge \omega^{j_q, \alpha-\beta}, \quad (\text{A.1})$$

we obtain the expansion of the MC equations for \mathcal{G} ,

$$\sum_{\alpha=s}^{\infty} \lambda^{\alpha} d\omega^{k_s, \alpha} = \sum_{\alpha=s}^{\infty} \lambda^{\alpha} \left[-\frac{1}{2} c_{i_p j_q}^{k_s} \sum_{\beta=0}^{\alpha} \omega^{i_p, \beta} \wedge \omega^{j_q, \alpha-\beta} \right], \quad (\text{A.2})$$

since the W-W conditions (4.3) will give zero in the r.h.s. unless $\alpha = p+q \geq s$, in agreement with the l.h.s. Eq. (A.2) can be made explicit for $p, q, s = 0, 1, \dots, n$ as follows:

$$\begin{aligned} \sum_{\alpha=s}^{\infty} \lambda^{\alpha} d\omega^{k_s, \alpha} = & -\frac{1}{2} \left[c_{i_0 j_0}^{k_s} \sum_{\alpha=0}^{\infty} \lambda^{\alpha} \sum_{\beta=0}^{\alpha} \omega^{i_0, \beta} \wedge \omega^{j_0, \alpha-\beta} + 2c_{i_0 j_1}^{k_s} \sum_{\alpha=1}^{\infty} \lambda^{\alpha} \sum_{\beta=0}^{\alpha-1} \omega^{i_0, \beta} \wedge \omega^{j_1, \alpha-\beta} + \dots \right. \\ & + 2c_{i_0 j_n}^{k_s} \sum_{\alpha=n}^{\infty} \lambda^{\alpha} \sum_{\beta=0}^{\alpha-n} \omega^{i_0, \beta} \wedge \omega^{j_n, \alpha-\beta} + c_{i_1 j_1}^{k_s} \sum_{\alpha=2}^{\infty} \lambda^{\alpha} \sum_{\beta=1}^{\alpha-1} \omega^{i_1, \beta} \wedge \omega^{j_1, \alpha-\beta} + \dots \\ & + 2c_{i_1 j_n}^{k_s} \sum_{\alpha=1+n}^{\infty} \lambda^{\alpha} \sum_{\beta=1}^{\alpha-n} \omega^{i_1, \beta} \wedge \omega^{j_n, \alpha-\beta} + \dots \\ & + c_{i_{n-1} j_{n-1}}^{k_s} \sum_{\alpha=2n-2}^{\infty} \lambda^{\alpha} \sum_{\beta=n-1}^{\alpha-n+1} \omega^{i_{n-1}, \beta} \wedge \omega^{j_{n-1}, \alpha-\beta} \\ & + 2c_{i_{n-1} j_n}^{k_s} \sum_{\alpha=2n-1}^{\infty} \lambda^{\alpha} \sum_{\beta=n-1}^{\alpha-n} \omega^{i_{n-1}, \beta} \wedge \omega^{j_n, \alpha-\beta} \\ & \left. + c_{i_n j_n}^{k_s} \sum_{\alpha=2n}^{\infty} \lambda^{\alpha} \sum_{\beta=n}^{\alpha-n} \omega^{i_n, \beta} \wedge \omega^{j_n, \alpha-\beta} \right]. \quad (\text{A.3}) \end{aligned}$$

Rearranging powers we get

$$\begin{aligned} \sum_{\alpha=s}^{\infty} \lambda^{\alpha} d\omega^{k_s, \alpha} = & -\frac{1}{2} \left\{ c_{i_0 j_0}^{k_s} \omega^{i_0, 0} \wedge \omega^{j_0, 0} + \lambda \left[c_{i_0 j_0}^{k_s} \sum_{\beta=0}^1 \omega^{i_0, \beta} \wedge \omega^{j_0, 1-\beta} + 2c_{i_0 j_1}^{k_s} \omega^{i_0, 0} \wedge \omega^{j_1, 1} \right] \right. \\ & + \lambda^2 \left[c_{i_0 j_0}^{k_s} \sum_{\beta=0}^2 \omega^{i_0, \beta} \wedge \omega^{j_0, 2-\beta} + 2c_{i_0 j_1}^{k_s} \sum_{\beta=0}^1 \omega^{i_0, \beta} \wedge \omega^{j_1, 2-\beta} \right. \\ & \left. \left. + 2c_{i_0 j_2}^{k_s} \omega^{i_0, 0} \wedge \omega^{j_2, 2} + c_{i_1 j_1}^{k_s} \omega^{i_1, 1} \wedge \omega^{j_1, 1} \right] + \dots \right\} . \end{aligned} \quad (\text{A.4})$$

Eq. (A.4) now gives

$$\begin{aligned} \sum_{\alpha=s}^{\infty} \lambda^{\alpha} d\omega^{k_s, \alpha} = & -\frac{1}{2} c_{i_0 j_0}^{k_s} \omega^{i_0, 0} \wedge \omega^{j_0, 0} \\ & - \sum_{\alpha=1}^{n-1} \lambda^{\alpha} \left[\frac{1}{2} \sum_{p=0}^{\lfloor \frac{\alpha}{2} \rfloor} c_{i_p j_p}^{k_s} \sum_{\beta=p}^{\alpha-p} \omega^{i_p, \beta} \wedge \omega^{j_p, \alpha-\beta} + \sum_{p=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \sum_{q=p+1}^{\alpha-p} c_{i_p j_q}^{k_s} \sum_{\beta=p}^{\alpha-q} \omega^{i_p, \beta} \wedge \omega^{j_q, \alpha-\beta} \right] \\ & - \sum_{\alpha=n}^{2n-1} \lambda^{\alpha} \left[\frac{1}{2} \sum_{p=0}^{\lfloor \frac{\alpha}{2} \rfloor} c_{i_p j_p}^{k_s} \sum_{\beta=p}^{\alpha-p} \omega^{i_p, \beta} \wedge \omega^{j_p, \alpha-\beta} + \sum_{p=0}^{\lfloor \frac{\alpha-1}{2} \rfloor} \sum_{q=p+1}^{\min\{\alpha-p, n\}} c_{i_p j_q}^{k_s} \sum_{\beta=p}^{\alpha-q} \omega^{i_p, \beta} \wedge \omega^{j_q, \alpha-\beta} \right] \\ & - \sum_{\alpha=2n}^{\infty} \lambda^{\alpha} \left[\frac{1}{2} \sum_{p=0}^n c_{i_p j_p}^{k_s} \sum_{\beta=p}^{\alpha-p} \omega^{i_p, \beta} \wedge \omega^{j_p, \alpha-\beta} + \sum_{p=0}^{n-1} \sum_{q=p+1}^n c_{i_p j_q}^{k_s} \sum_{\beta=p}^{\alpha-q} \omega^{i_p, \beta} \wedge \omega^{j_q, \alpha-\beta} \right] , \end{aligned} \quad (\text{A.5})$$

that is

$$\begin{aligned} \sum_{\alpha=s}^{\infty} \lambda^{\alpha} d\omega^{k_s, \alpha} = & -\frac{1}{2} c_{i_0 j_0}^{k_s} \omega^{i_0, 0} \wedge \omega^{j_0, 0} - \sum_{\alpha=1}^{\infty} \lambda^{\alpha} \left[\frac{1}{2} \sum_{p=0}^{\min\{\lfloor \frac{\alpha}{2} \rfloor, n\}} c_{i_p j_p}^{k_s} \sum_{\beta=p}^{\alpha-p} \omega^{i_p, \beta} \wedge \omega^{j_p, \alpha-\beta} + \right. \\ & \left. + \sum_{p=0}^{\min\{\lfloor \frac{\alpha-1}{2} \rfloor, n-1\}} \sum_{q=p+1}^{\min\{\alpha-p, n\}} c_{i_p j_q}^{k_s} \sum_{\beta=p}^{\alpha-q} \omega^{i_p, \beta} \wedge \omega^{j_q, \alpha-\beta} \right] , \end{aligned} \quad (\text{A.6})$$

from which we obtain, upon explicit imposition of the contraction condition (4.3) on the structure constants c 's:

$\alpha = s = 0$:

$$d\omega^{k_0, 0} = -\frac{1}{2} c_{i_0 j_0}^{k_0} \omega^{i_0, 0} \wedge \omega^{j_0, 0} ; \quad (\text{A.7})$$

$\alpha = s \geq 1$, s odd:

$$d\omega^{k_s, s} = - \sum_{p=0}^{\frac{s-1}{2}} c_{i_p j_{s-p}}^{k_s} \omega^{i_p, p} \wedge \omega^{j_{s-p}, s-p} ; \quad (\text{A.8})$$

$\alpha = s \geq 1$, s even:

$$d\omega^{k_s, s} = -\frac{1}{2} c_{i_{\frac{s}{2}} j_{\frac{s}{2}}}^{k_s} \omega^{i_{\frac{s}{2}}, \frac{s}{2}} \wedge \omega^{j_{\frac{s}{2}}, \frac{s}{2}} - \sum_{p=0}^{\frac{s-2}{2}} c_{i_p j_{s-p}}^{k_s} \omega^{i_p, p} \wedge \omega^{j_{s-p}, s-p} ; \quad (\text{A.9})$$

$\alpha > s \geq 0$:

$$\begin{aligned}
d\omega^{k_s, \alpha} = & -\frac{1}{2} \sum_{p=\lceil \frac{s+1}{2} \rceil}^{\min\{\lceil \frac{\alpha}{2} \rceil, n\}} c_{i_p j_p}^{k_s} \sum_{\beta=p}^{\alpha-p} \omega^{i_p, \beta} \wedge \omega^{j_p, \alpha-\beta} \\
& - \sum_{p=0}^{\min\{\lceil \frac{\alpha-1}{2} \rceil, n-1\}} \sum_{q=\max\{s-p, p+1\}}^{\min\{\alpha-p, n\}} c_{i_p j_q}^{k_s} \sum_{\beta=p}^{\alpha-q} \omega^{i_p, \beta} \wedge \omega^{j_q, \alpha-\beta} .
\end{aligned} \tag{A.10}$$

References

- [1] F.J. Dyson (ed.), *Symmetry groups in nuclear and particle physics*, W.A. Benjamin, N.Y. (1966)
- [2] S. Ferrara (ed.), *Supersymmetry* (vols. I,II), World Sci. (1987)
- [3] I.E. Segal, *A class of operator algebras which are determined by groups*, Duke Math. J. **18**, 221-265 (1951)
- [4] E. İnönü and E.P. Wigner, *On the contraction of groups and their representations*, Proc. Nat. Acad. Sci. U.S.A. **39**, 510-524 (1953); E. İnönü, *contractions of Lie groups and their representations*, in *Group theoretical concepts in elementary particle physics*, F.Gürsey ed., Gordon and Breach, pp. 391-402 (1964)
- [5] E.J. Saletan, *Contractions of Lie groups*, J.Math. Phys. **2**, 1-21 (1961)
- [6] D. Arnal and J.-C. Cortet, *Contractions and group representations*, J. Math. Phys. **20**, 556-563 (1979)
- [7] E. Celeghini and M. Tarlini, *Contractions of group representations*, Nuov. Cim. **61B**, 265-277, 172-180 (1981), *ibid.* **68B**, 133-141 (1982)
- [8] E.A. Lord, *Geometrical interpretation of İnönü-Wigner contractions*, Int. J. Theor. Phys. **24**, 723-730 (1985)
- [9] M. de Montigny and J. Patera, *Discrete and continuous graded contractions of Lie algebras and superalgebras*, J. Phys. **A24**, 525-547 (1991); R.V. Moody and J. Patera, *Discrete and continuous graded contractions of representations of Lie algebras*, J. Phys. **A24**, 2227-2257 (1991).
- [10] F. Herranz, M. De Montigny, M.A. del Olmo and M. Santander, *Cayley-Klein algebras as graded contractions of $so(N+1)$* , J. Phys. **A27**, 2515-2526 (1994)
- [11] E. Weimar-Woods, *Contractions of Lie algebras: generalized İnönü-Wigner contractions versus graded contractions*, J. Math. Phys. **36**, 4519-4548 (1995); *Contractions, generalized İnönü and Wigner contractions and deformations of finite-dimensional Lie algebras*, Rev. Math. Phys. **12**, 1505-1529 (2000)
- [12] E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, *Three dimensional quantum groups from contractions of $su(2)_q$* , J. Math. Phys. **31**, 2548-2551 (1990); *Contractions of quantum groups*, Lect. Notes in Math. **1510**, p. 221, Springer (1992)

- [13] M. Gerstenhaber, *On the deformations of rings and algebras*, Ann. Math. **79**, 59-103 (1964)
- [14] A. Nijenhuis and R.W. Richardson Jr., *Cohomology and deformations in graded Lie algebras*, Bull. A. Math. Soc. **72**, 1-29 (1966)
- [15] A. Nijenhuis and R.W. Richardson Jr., *Deformations of Lie algebra structures*, J. Math. Mech. **171**, 89-105 (1967)
- [16] R.W. Richardson, *On the rigidity of semi-direct products of Lie algebras*, Pac. J. Math. **22**, 339-344 (1967)
- [17] M. Lévy-Nahas, J. Math. Phys., *Deformation and contraction of Lie algebras*, J. Math. Phys. **8**, 1211-1222 (1967)
- [18] R. Hermann, *Analytic continuation of group representations III*, Commun. Math. Phys. **3**, 75-97 (1966); *Vector bundles in mathematical physics* (vol. II), W. A. Benjamin (N.Y.), 1970, p. 107
- [19] R. Gilmore, *Rank 1 expansions*, J. Math. Phys. **13**, 883-886 (1972)
- [20] J. Moyal, *Quantum mechanics as a statistical theory*, Proc. Camb. Phil. Soc. **45** 99-124 (1949)
- [21] M. Flato, A. Lichnerowicz and D. Sternheimer, *Deformations of Poisson brackets, Dirac brackets and applications*, J. Math. Phys. **17**, 1754-1762 (1976); F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantization*, Ann. Phys. **111**, 61-110, 111-151 (1978)
- [22] J. Vey, *Déformation du crochet de Poisson sur une variété symplectique*, Comment. Math. Helvetici **50**, 421-454 (1975)
- [23] J.A. de Azcárraga and J.M. Izquierdo, *Lie groups, Lie algebras, cohomology and some applications in physics*, Camb. Univ. Press., 1995
- [24] V. Aldaya and J.A. de Azcárraga, *Cohomology, central extensions and dynamical groups*, Int. J. Theor. Phys. **24**, 141-154 (1985)
- [25] D.A. Leites, *Cohomology of Lie superalgebras*, Funk. Analiz. **9**, 75-76 (1975)
- [26] D'Auria, P. Fré and T. Regge, *Graded Lie algebra, cohomology and supergravity*, Riv. Nuovo Cim. **3**, fasc. 12 (1980)
- [27] L. Castellani, R. D'Auria and P. Fré, *Supergravity and superstrings: a geometric perspective* (vols. I,II), World Scientific (1991)
- [28] M. Scheunert and R.B. Zhang, *Cohomology of Lie superalgebras and of their generalizations*, J. Math. Phys. **39**, 5024-5061 (1998), q-alg/9701037
- [29] B. Binetgar, *Cohomology and deformation of Lie superalgebras*, Lett. Math. Phys. **12**, 301-308 (1986)

- [30] D. Alekseevsky, P.W. Michor and W. Ruppert, *Extensions of super Lie algebras*, math.QA/0101190, unpublished
- [31] S. Ferrara and M.A. Lledó, *Considerations on super Poincaré algebras and their extensions to simple superalgebras*, Rev. Math. Phys. **14**, 519-530 (2002), hep-th/0112177
- [32] C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Am. Math. Soc. **63**, 85-124 (1948)
- [33] J.A. de Azcárraga and P. Townsend, *Superspace geometry and the formulation of supersymmetric extended objects*, Phys. Rev. Lett. **62**, 2579-2582 (1989)
- [34] C. Chryssomalakos, J.A. de Azcárraga, J.M. Izquierdo and J.C. Pérez Bueno, *The geometry of branes and extended superspaces*, Nucl. Phys. **B567**, 293-330 (2000), hep-th/9904137
- [35] P. Townsend, *M-theory from its superalgebra*, Cargèse lectures, hep-th/9712004
- [36] R. D'Auria and P. Fré, *A geometric supergravity and its hidden supergroup*, Nucl. Phys. **B201**, 101-140 (1982) [E.: *ibid.*, **B206**, 496 (1982)]
- [37] W. van Holten and P. van Proeyen, *N=1 supersymmetry algebras in d=2,3,4 mod 8*, J. Phys. **15**, 3763-3783 (1982)
- [38] I. Bars, *S-theory*, Phys. Rev. **D55**, 2373-2381 (1997), hep-th/9607112
- [39] E. Sezgin, *The M-algebra*, Phys. Lett. **392**, 323-331 (1997), hep-th/9609086
- [40] C. Chryssomalakos, *Stability of Lie superalgebras and branes*, Mod. Phys. Lett. **A16**, 197-210 (2001), hep-th/0102134
- [41] V. Aldaya and J.A. de Azcárraga, *A note on covariant derivatives in supersymmetry*, J. Math. Phys. **26**, 1818-1821 (1985)
- [42] M. Hatsuda and M. Sakaguchi, *Wess-Zumino term for the AdS superstring and generalized İnönü-Wigner contraction*, hep-th/0106114
- [43] D. Sullivan, *Infinitesimal computations in topology*, Inst. des Haut. Étud. Sci., Pub. Math **47**, 269-331 (1977)
- [44] P. van Nieuwenhuizen, *Free graded differential superalgebras*, in *Group theoretical methods in physics*, M. Serdaroglu and E. İnönü Eds., Lect. Notes in Phys. **180**, 228-247 (1983)
- [45] M. Hatsuda and M. Sakaguchi, *Wess-Zumino terms for the AdS superstring*, Phys. Rev. **D66**, 045020 (2002), hep-th/0205092
- [46] R. R. Metsaev and A. A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, Nucl. Phys. **533**, 109-126 (1998), hep-th/9805028
- [47] I. Bandos, J.A. de Azcárraga, J.M. Izquierdo and J. Lukierski, *BPS states in M theory and twistorial constituents*, Phys. Rev. Letts. **86**, 4451-4454 (2001)

- [48] E. Bergshoeff and E. Sezgin, *Super p -brane theories and new space-time superalgebras*, Phys. Lett. **B354**, 256-263 (1995), hep-th/9504140
- [49] M.B. Green, *Supertranslations, superstrings and Chern-Simons forms*, Phys. Lett. **B223**, 157-164 (1989)
- [50] S. Deser, R. Jackiw and S. Templeton, *Three-dimensional massive theories*, Phys. Rev. Lett. **48**, 975-978 (1982); *Topologically massive gauge theories*, Ann. Phys. **140**, 372 (1982) [E.: *ibid.* **185**, 406 (1988)]
- [51] J. Zanelli, *Quantization of the gravitational constant in odd dimensions*, Phys. Rev. **D51**, 490-492 (1995), hep-th/9406202
- [52] P. van Nieuwenhuizen, *Three-dimensional conformal supergravity and Chern-Simons terms*, Phys. Rev. **D32**, 872-878 (1985)
- [53] A. Achúcarro, P.K. Townsend, *A Chern-Simons action for three-dimensional anti-de Sitter supergravity theories*, Phys. Lett. **B180**, 89-92 (1986); *Extended supergravities in $d=2+1$ as Chern-Simons theories*, Phys. Lett. **B229**, 383-387 (1989)
- [54] E. Witten, *$(2+1)$ -dimensional gravity as an exactly soluble system*, Nucl. Phys. **B311**, 46 (1988)
- [55] S.J. Gates, M.T. Grisaru M. Roček and W. Siegel, *Superspace: 1001 lessons in supersymmetry*, Benjamin/Cummings (1983)
- [56] D. Cangemi and R. Jackiw, *Gauge invariant formulation of lineal gravity*, Phys. Rev. Lett. **69**, 233-236 (1992), hep-th/9203056; R. Jackiw, *Higher symmetries in lower dimensional models*, in Proc. of the *XXIII GIFT Seminar* (Salamanca, 1992), L. A. Ibort and M. A. Rodríguez eds., NATO ASI Series **409**, 289-316 (1993)
- [57] V.O. Rivelles, *Topological two-dimensional dilaton supergravity*, Phys. Lett **B321**, 189-192 (1994), hep-th/9301029
- [58] P. Hořava, *M theory as a holographic theory*, Phys. Rev. **D59**, 046004-1-11 (1999), hep-th/9712130
- [59] A.H. Chamseddine, *Topological gauge theory of gravity in five and all odd dimensions*, Phys. Lett. **233B**, 291-294 (1989); *Topological gravity and supergravity in various dimensions*, Nucl. Phys. **B346**, 213-234 (1990);
- [60] R. Troncoso and J. Zanelli, *New gauge supergravity in seven and eleven dimensions*, Phys. Rev. **D58**, 101703 (1998), hep-th/9710180; J. Zanelli, *Chern-Simons gravity: from $2+1$ to $2n+1$ dimensions*, hep-th/0010049
- [61] M. Bañados, *The linear spectrum of $OSp(32|1)$ Chern-Simons supergravity in eleven dimensions*, Phys. Rev. Lett. **88**, 031301 (2002), hep-th/0107214
- [62] P. Mora and H. Nishino, *Fundamental extended objects for Chern-Simons supergravity*, Phys. Lett. **B482**, 222-232 (2000), hep-th/0002077